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PLANE INVOLUTIONS OF ORDER FOUR.

By Temple Rice Hollcroft.

1. A plane involution of order four or a (1, 4) point correspondence between two planes is said to exist when to one point in the first plane correspond four points in the second to each of which corresponds the original point in the first. The planes may be distinct or coincident.

Plane involutions of order two have been completely classified by Bertini* and those of order three by Miss Howe.† Miss Scott‡ has treated metrically a case of (1, 4) point correspondence which is a special form of type 3 of this classification.

The purpose of this paper is to discuss and classify the algebraic methods of relating two planes in (1, 4) point correspondence. All the methods that are reducible to each other by means of birational transformations will be counted as one, and the most general of these will be called a type of (1, 4) point correspondence. The types are thus birationally independent.

The general methods employed are due to Sharpe and Snyder§ and have been made use of since by A. M. Howe.|| The new features of plane involutions of order four over those of order three and two are found principally in the discussion of coincident images. Involutions of order higher than four, however, present no new features, that is, an involution of order four may be generalized for any order n as demonstrated in the latter part of this paper. It is felt, therefore, that involutions of order four are of sufficient importance to warrant their classification, since they are of the highest order that has distinctive characteristics and those of order n cannot be classified.

2. A type is defined by two algebraic equations in x_1, x_2, x_3 and x_1', x_2', x_3' of the form

$$x_1'u_1 + x_2'u_2 + x_3'u_3 = 0, \tag{1}$$

$$x_1'v_1 + x_2'v_2 + x_3'v_3 = 0, \tag{2}$$

* E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. VIII (1877), pp. 244–286.

† A. M. Howe, "A Classification of Plane Involutions of Order Three," American Journal of Math., Vol. XLI (1919), pp. 25–40.

‡ C. A. Scott, "Studies in Transformation of Plane Algebraic Curves," *Quarterly Journal of Mathematics*, Vol. 29 (1899), pp. 329–381, and Vol. 32 (1901), pp. 209–239.

§ F. R. Sharpe and V. Snyder, "Types of (2, 2) point correspondences between two planes," *T. A. M. S.*, Vol. 18 (1917), pp. 402–414.

|| A. M. Howe, loc. cit.

wherein $u_i = 0$ and $v_i = 0$ are curves of (x) intersecting in four variable points. These equations so relate the planes (x) and (x') that to a point P' of (x') correspond P_1, P_2, P_3, P_4 of (x) and to each of these image points of (x) corresponds the original point P', for the coördinates of a point of (x') determine uniquely a pair of curves of (x) intersecting in four non-basic points and the two curves so determined are in (1, 1) correspondence with the coördinates of the point of (x'). Ordinarily the four image points will be distinct, but the coördinates of P' may be so chosen that the corresponding curves of (x) shall have one or two contacts. The locus of such points P' is a curve of (x') called the curve of branch points. It will be denoted by the letter L'. The locus of the contacts in (x) is called the coincidence curve, denoted by K. When two image points coincide on K, the other two lie on a residual curve Γ. The complete image of K is L'; of L', $K^2\Gamma$; of Γ, L'^2. L' and K are in (1, 1) correspondence; L' and Γ (and therefore K and Γ) are in (1, 2) point correspondence.

To a line of (x') corresponds a curve of (x) whose image is the original line of (x') counted four times. To two lines of (x') intersecting at P' correspond two curves of (x) intersecting in four non-basic points, the images of P'. The remaining intersections of the two image curves of (x) are fixed points common to all image curves of (x). Since these curves are in (1, 1) correspondence with the lines of (x') they form a net. The basis points of this net are the fixed points common to all the image curves. Their images in (x') are basis curves the order of the curve being equal to the multiplicity of the point on the line images. When a given curve of (x) passes through a basis point, its image in (x') is composite, consisting of the curve-image of the basis point (counted as many times as the multiplicity of the basis point on the given curve) and a curve called the proper image of the given curve of (x).

The jacobian of the net of line image curves of (x) is the locus of the contacts of the curves of the net. It therefore either is the coincidence curve K or contains it as a factor. In the latter case the other factors are basis curves of (x). The residual curve Γ is sometimes called the cojacobian.

To the lines of (x) correspond rational curves of (x'). Since the transformation from (x') to (x) is not rational the image curves of (x') can have basis points only of higher multiplicity than two. The images of these basis points are irrational basis curves of (x) whose order is the multiplicity of the basis point on the line images of (x').

3. There are ten independent types of (1, 4) point correspondences. Each type is defined by two equations of the form described in the preceding section. The equations are both linear in (x'), so the types depend only

on the choice of u_i and v_i of (x). The following table shows the curves represented by u_i and v_i for each type. Subscripts of C denote the order of a curve, coefficients and subscripts of P the number and multiplicity respectively of its basis points.

Type	$u_i = 0$	$v_i = 0$
1	line pencil	C_n; P_{n-4}
2	line	quartic
3	cubic; $5P_1$	cubic; $5P_1$
4	conic	conic
5	cubic; $8P_1$	C_{12}; $8P_4$
6	cubic; $8P_1$	quartic; $8P_1$
7	conic; P_1	cubic; P_2
8	conic; $2P_1$	cubic; $2P_1$
9	conic; $2P_1$	quartic; $2P_2$
10	cubic; $7P_1$	quartic; P_2, $6P_1$

4. *Type 1.*—The defining equations are

$$x_1x_1' + x_2x_2' = 0, \tag{1}$$

$$x_1'\varphi_1 + x_2'\varphi_2 + x_3'\varphi_3 = 0, \tag{2}$$

$\varphi_i(x) \equiv x_3^4u_i(x_1, x_2) + x_3^3v_i(x_1, x_2) + x_3^2w_i(x_1, x_2) + x_3s_i(x_1, x_2) + t_i(x_1, x_2) = 0$, wherein u_i, v_i, w_i, s_i, t_i are homogeneous functions of the respective degrees $n-4$, $n-3$, $n-2$, $n-1$, n in x_1, x_2. Then equation (2) represents a C_n with an $(n-4)$-fold point at $P \equiv (0, 0, 1)$, the vertex of the line pencil. The equations of transformation from (x') to (x) are

$$\rho x_1' = -x_2\varphi_3,$$
$$\rho x_2' = x_1\varphi_3,$$
$$\rho x_3' = x_2\varphi_1 - x_1\varphi_2.$$

A line of (x') with coefficients a_1', a_2', a_3' corresponds to

$$C_{n+1} \equiv (a_1'x_2 - a_2'x_1)\varphi_3 + a_3'(x_1\varphi_2 - x_2\varphi_1) = 0.$$

C_{n+1} has P_{n-3} and is of genus $3n-6$. The curves $\varphi_3 = 0$ and $x_1\varphi_2 - x_2\varphi_1 = 0$ intersect in $8n-12$ points outside of P. These $8n-12$ P_1 are simple basis points of (x) through which pass all image curves of (x). Two line image curves intersect in four non-basic points which are collinear with P and are the images of the intersection of the two lines of (x'). If the line of (x') passes through $P' \equiv (0, 0, 1)$, its image is

$$\varphi_3(a_1'x_2 - a_2'x_1) = 0.$$

The curve $\varphi_3 = 0$ is fundamental, the image of P', and the other factor is the proper image of the line through P'.

To a line of (x) with coefficients a_1, a_2, a_3 corresponds

$$C'_{n+1} \equiv x'_1\varphi'_1 + x'_2\varphi'_2 + x'_3\varphi'_3 = 0,$$

wherein φ'_i has the same form as φ_i with $-a_3x'_2$, $a_3x'_1$ and $a_1x'_2 - a_2x'_1$ substituted respectively for x_1, x_2 and x_3. C'_{n+1} has an n-fold point at P' and is of genus 0. The proper image of a line through P is a line through P' counted four times. The basis point P has for its image the basis curve

$$u' \equiv x'_1u_1(-x'_2, x'_1) + x'_2u_2(-x'_2, x'_1) + x'_3u_3(-x'_2, x'_1) = 0.$$

The basis curve u' is of order $n-3$ with P'_{n-4}.

Each of the $8n-12$ P_1 corresponds to a line of (x') through P' and tangent to L'.

5. The equation of the branch-point curve L' is the condition on the parameters x'_1 that the line and C_n given by the defining equations have a point of contact. This condition is of degree $6n-12$ in the coefficients of the line and 6 in the coefficients of the C_n. L' is therefore of order $6n-6$ with P'_{6n-12}. L' will also be determined as the image of K.

The jacobian of the net of line images in (x) is

$$\varphi_3\left[\varphi_3\left(x_2\frac{\partial\varphi_1}{\partial x_3} - x_1\frac{\partial\varphi_2}{\partial x_3}\right) - \frac{\partial\varphi_3}{\partial x_3}(x_2\varphi_1 - x_1\varphi_2)\right] = 0.$$

The jacobian consists of the basis curve $\varphi_3 = 0$ and the coincidence curve K. K is of order $2n$ and genus $10n-20$ with P_{2n-6} and $8n-12$ P_1. The image of K is L'_{6n-6} with P'_{6n-12} and of genus $10n-20$. Then L' has the equivalent of $20n-30$ double points.

If in the two defining equations we solve the equations of the line and C_n simultaneously, simplify and arrange in powers of x_3/x_1, the discriminant of the quartic in x_3/x_1 is the equation of L'. Thus

$$L'_{6n-6} \equiv S^3 + 27T^2,$$

wherein

$$S \equiv 4s'v' - u't' - 3w'^2,$$
$$T \equiv u'w't' + 2v'w's' - v'^2t' - u's'^2 - w'^3;$$

u' is the fundamental curve of (x') and v', w', s', t' are curves whose equations are like that of u' with v, w, s, t substituted respectively for u. The curves S and T intersect in $12n-18$ non-basic points, which are cusps of L'. Then of the $20n-30$ double points of L' $12n-18$ are cusps and $8n-12$, nodes. The coördinates of a cusp of L' determine the line and C_n of the defining equations so that the line is tangent to the C_n at a point of inflection, and the coördinates of a double point of L' so that the line is a double tangent to the C_n.

The class of L' is $20n-36$. Then from P' $8n-12$ tangents can be drawn to L' which are the basis lines of (x').

The image of L' is a curve of order $12n - 6$ consisting of K counted twice and the residual curve Γ_{8n-6} which has P_{8n-18}, $8n - 12P_4$, $12n - 18$ cusps and is of genus $28n - 53$.

K and Γ have $40n - 60$ intersections among which are $12n - 18$ contacts, the common tangent passing through P. These contacts correspond to the cusps of L', that is, at these contacts three image points coincide. The fourth image point lies at a cusp of Γ on this common tangent. The remaining $16n - 24$ are simple intersections, images of the nodes of L'. To a node of L' correspond two intersections of K and Γ such that the line joining them is tangent to Γ at both points and passes through P. The two intersections represent a pair of coincidences.

The class of K is $24n - 42$. From P $20n - 30$ tangents can be drawn to K consisting of $8n - 12$ at the simple basis points and $12n - 18$ at the contacts of K and Γ. The class of Γ is $60n - 102$. From P $44n - 66$ tangents may be drawn to Γ which consist of $12n - 18$ at the contacts of K and Γ, $8n - 12$ bitangents through P and the $8n - 12$ P_1 and $8n - 12$ bitangents at the pairs of coincidences.

The four images of a point of (x') cannot ordinarily coincide because a general C_n of the defining equations has no point of undulation. When the equation of the C_n is so chosen that it will have such a point, certain of the pairs of coincidences will themselves coincide making four coincident images. Such a point is an undulation of Γ, the double tangent passing through P and a node of K neither of whose branches has contact with Γ. The corresponding point of L' is a cusp of the second kind.

6. L' and the basis curve u' intersect in $12n - 30$ points of which $2n - 6$ are contacts corresponding to the $2n - 6$ directions of K through P and $8n - 18$ intersections corresponding to the directions of Γ through P.

The image of the basis curve u' is P_{n-3} and a residual curve p of order $2n - 3$ with P_{2n-6} and $8n - 12$ P_1. K and p have the same tangents at P. K meets p in $10n - 24$ points besides at basis points. Of these $2n - 6$ lie on the $2n - 6$ common tangents to K and p at P and $8n - 18$ lie on the $8n - 18$ tangents to Γ at P. The class of p is $12n - 30$. From P $8n - 18$ tangents can be drawn to p. These tangents coincide with the tangents to Γ at P and pass through intersections of p with K and of p with Γ. Γ and p intersect in $16n - 42$ points of which $8n - 24$ are $4n - 12$ contacts lying two each on the $2n - 6$ common tangents to p and K at P and $8n - 18$ intersections lying on the tangents to p from P and to Γ at P. The complete image of p is u' counted three times.

The image of the point of contact of a basis line of (x') and L' consists of two consecutive points of K at the basis point and two pairs of consecutive points of Γ not at the basis point. Then the line joining P to

a P_1 has a contact with K at P_1 and two contacts with Γ not at P_1. A line through P and a P_1 has for its image the basis line of (x') corresponding to that P_1 counted three times.

7. For the remaining types only the most important results are given. The defining equations for each type may be constructed from the table in section 3. The following notation will be used in tabulating results:

The symbol $\sim$ meaning "corresponds to."

L', K, Γ, curves as heretofore described.

C, C', variable curves of (x) and (x').

P, P', basis points of (x) and (x').

$\overline{P}$, $\overline{P}'$, variable points of (x) and (x').

k, double points of L'.

Subscripts of curves denote their order.

Subscripts of points denote their multiplicity on the curve being described.

Type 2.

$C'_1 \sim C_5, 21P_1$.

$C_1 \sim C'_5, \overline{P}'_4$.

$L'_{18}, 102k \sim K^2_{12}, 21P_2; \Gamma_{66}, 21P_{14}$.

Type 3.

$C'_1 \sim C_6, 5P_2, 12P_1$.

$C_1 \sim C'_6, 10\overline{P}'_2$.

$L'_{16}, 76k \sim K^2_{15}, 5P_5, 12P_2; \Gamma_{66}, 5P_{22}, 12P_{12}$.

Type 4.

$C'_1 \sim C_4, 12P_1$.

$C_1 \sim C'_4, 3\overline{P}'_2$.

$L'_{12}, 39k \sim K^2_9, 12P_2; \Gamma_{30}, 12P_8$.

Type 5. Q_1 is the ninth basis point of the cubic pencil.

$C'_1 \sim C_{15}, 8P_5, Q_1, 20P_1$.

$C_1 \sim C'_{15}, P'_{12}, 25\overline{P}'_2$.

$Q_1 \sim$ line not through P'.

$P_1 \sim$ line through P'.

$L'_{28}, P'_{20}, 116k \sim K^2_{30}, 8P_{10}, Q_2, 20P_1; \Gamma_{120}, 8P_{40}, Q_{24}, 20P_6$.

Type 6.

$C'_1 \sim C_7, 8P_2, Q_1, 12P_1$.

$C_1 \sim C'_7, P'_4, 9\overline{P}'_2$.

$L'_{20}, P'_{12}, 79k \sim K^2_{14}, 8P_4, Q_2, 12P_1; \Gamma_{64}, 8P_{20}, Q_{16}, 12P_6$.

Type 7.

$C'_1 \sim C_5, P_3, 12P_1$.

$C_1 \sim C'_5, \overline{P}'_4$.

$L'_{12}, 40k \sim K^2_{12}, P_8, 12P_2; \Gamma_{36}, P_{20}, 12P_8$.

Type 8.

$C_1' \sim C_5, 2P_2, 13P_1.$

$C_1 \sim C_5', 6\overline{P}_2'.$

$L_{14}', 56k \sim K_{12}^2, 2P_5, 13P_2; \Gamma_{46}, 2P_{18}, 13P_{10}.$

Type 9.

$C_1' \sim C_6, 2P_3, 14P_1.$

$C_1 \sim C_6', 10\overline{P}_2'.$

$L_{14}', 57k \sim K_{15}^2, 2P_8', 14P_2; \Gamma_{54}, 2P_{26}, 14P_{10}.$

Type 10.

$C_1' \sim C_7, P_3, 6P_2, 12P_1.$

$C_1 \sim C_7', 15\overline{P}_2'.$

$L_{18}', 100k \sim K_{18}^2, P_8, 6P_5, 12P_2; \Gamma_{90}, P_{38}, 6P_{26}, 12P_{14}.$

8. Any two nets of curves having four variable intersections can be reduced by a series of quadratic transformations to a pair of nets of one of the ten types given in section 3. This method is the same as that used by Bertini* in showing that all pairs of nets intersecting in two variable points are birationally equivalent to a pair of nets of one of the three types of plane involutions of order two.

As an illustration of this method, consider a net of cubics with $6P_1$ and a net of quartics with $2P_2$, $4P_1$ at the basis points of the cubics. Choosing the $2P_2$ and any P_1 as the vertices of the triangle of inversion, the quartic net corresponds to a net of cubics with $5P_1$. The net of cubics with $6P_1$ is invariant, so the above system reduces to two nets of cubics with five common basis points, which is type 3.

A net of quintics with $16P_1$ at the basis points of a quartic pencil reduces to type 2 because the four remaining intersections of the quintic and a quartic of the pencil are collinear.

Two nets of quartics with 12 common basis points reduce to type 4 since the lines and points of (x') are in (1, 1) correspondence.

For plane involutions above the fourth order the number of types increases very rapidly. There is no relation between the order of the involution and the number of types except that one increases with the other.

9. Types 1, 2 and 5 are capable of generalization for any n. The results for type 1 will be given. In the defining equations written as in type 1 the φ_i now represent curves of order $m \geq n$ with P of multiplicity $m - n$.

$C_1' \sim C_{m+1}, P_{m-n+1}, n(2m - n + 1)P_1$, genus $(n - 1)(m - n/2)$.

$C_1 \sim C_{m+1}', P_m'.$

$L_{(n-1)(2m-n+2)}', P_{(n-1)(2m-n)}', 3(n - 2)(2m - n + 1)$ cusps,

* E. Bertini, loc. cit.

$2(n-2)(n-3)(2m-n+1)$ nodes, genus $(2m-n)(2n-3)$.

K_{2m}, $P_{2(m-n+1)}$, $n(2m-n+1)P_1$, genus $(2m-n)(2n-3)$.

$\Gamma_{(n-2)(4m-n+1)}$, $P_{(n-2)(4m-3n+3)}$, $n(2m-n+1)P_{2(n-2)}$,

$3(n-2)(n-3)(2m-n+1)$ cusps, genus

$$(n-2)[(2m-n+1)(2n^2-10n+17)-4m]+1.$$

For $n > 4$ nothing new presents itself for the most general form of the defining equations since a general curve does not have tangents of higher multiplicity than two. So for any integer n in general there can be no more than two or three coincident images or a pair of coincidences. But the C_m of the defining equations may be so chosen that it will have any number of consecutive points or groups of consecutive points all collinear up to n and in these cases points of (x') lying at singular points of L' can be found whose coördinates cause this multiple tangent to pass through P and as many of whose images will coincide and in the same manner as this tangent has consecutive points of C_m. Similar conclusions apply to all types of plane involutions of order n.

10. For $m = n$ the $(1, n)$ Jonquières transformation (type 1 for $n = 4$) can be represented geometrically by means of a surface S of order $n+1$ with a simple point P_1 and an n-fold point P_n through which pass $n(n+1)$ lines of the surface.

Choose any two planes (x) and (x') as the simple and n-fold planes respectively. The line joining any point P_i' of (x') to P_1 cuts S in n points which project from P_n on (x) into n collinear points, the images in (x) of P_i'. The intersections of the line joining P_1 to P_n with (x) and (x') give the basis points P and P' respectively. Since the n intersections of $P_i'P_1$ with S lie in a plane determined by P_1P_n and $P_i'P_1$, the n image points of (x) are collinear with P.

Likewise to a point P_i of (x) corresponds one point in (x') found by projecting the single intersection of P_nP_i with S from P_1 on the plane (x').

The intersections of the $n(n+1)$ lines of S through P_n with (x) determine the $n(n+1)$ simple basis points of (x). The projection of these lines from P_1 on (x') gives the $n(n+1)$ basis lines of (x') all passing through P'.

Choose a line C_1' of (x') not through P'. A plane through C_1' and P_1 cuts S in a curve of order $n+1$ with a simple point at P and $n(n+1)$ fixed points on the lines of S. The projection of this curve on (x) from P_n gives C_{n+1}, the image of C_1' with a simple point at P and $n(n+1)$ simple basis points. To two lines of (x') intersecting in P_i' correspond two image curves of (x) intersecting in the n images of P_i' all collinear with P.

When C_1' passes through P' the plane determined by C_1' and P_1 passes through P_n and the projection of the curve common to that plane and S from P_n on (x) is a line through P cut out by that plane. Thus the pencil of planes on P_1P_n cuts out the corresponding rays of the pencils on P and P' of (x) and (x') respectively. The $n(n+1)$ fixed planes of the pencil determined by the $n(n+1)$ lines of S cut out the $n(n+1)$ basis lines of (x') and the lines of (x) through P and the respective P_1. Only proper images appear in this construction.

A plane through a line C_1 of (x) and P_n cuts S in a curve of order $n+1$ with an n-fold point at P_n which projects from P_1 into a C_{n+1}' of (x') with P' of multiplicity n.

The tangent cone to S from P_1 is of order $(n-1)(n-2)$. From P_1 can be drawn a certain number of inflectional tangents and double tangents which are respectively cuspidal and double edges of the tangent cone. The intersection of this tangent cone with (x') gives the branch-point curve L' whose cusps and nodes are occasioned by the cuspidal and double edges of the cone. The basis lines of (x') are seen to be tangents to L'.

The projection from P_n on (x) of the contour curve common to S and the tangent cone gives the coincidence curve K. Each element of the tangent cone cuts S in $n-2$ other points besides at the ordinary contacts. The projection on (x) from P_n of these residual intersections is the curve Γ. An inflectional contact projects from P_n into three consecutive points, two on K and one on Γ, so that K and Γ touch at that point and their common tangent passes through P since it is the projection of the inflectional tangent to S. The projection of the $n-3$ residual intersections of the cuspidal edge of the tangent cone with S gives $n-3$ cusps of Γ all lying on the common tangent to K and Γ. In the projection of a double contact from P_n each contact goes into a point of K and two consecutive points of Γ so that the projection of the bitangent to S is a bitangent to Γ through P. It is not tangent to K.

Each of the lines through P and a simple basis point of (x) is tangent to K at that simple basis point and has $n-2$ contacts with Γ elsewhere.

For $n=1$ the preceding becomes a three-dimensional construction for the ordinary quadratic transformation.

WELLS COLLEGE,
April, 1921.

DIFFERENTIAL EQUATIONS WITH A CONTINUOUS INFINITUDE OF VARIABLES.

By I. A. Barnett.

The existence of solutions of the simultaneous system of differential equations

$$\frac{du_\rho}{d\tau} = f_\rho(\tau, u_1, \cdots u_\nu), \qquad (\rho = 1 \cdots \nu)$$

has been proved by means of various devices. Moreover the nature of the solutions when regarded as functions of the initial constants has also been treated.*

Some of these results have been extended to differential equations with a denumerable infinitude of variables by Von Koch, Moulton and Hart.†

In 1911, G. Kowalewski discussed the existence theorem for differential equations which involved Schmidt integral power series,‡ and in 1914, Volterra stated an existence theorem for general differential equations involving functions of lines.§ He gives, however, few details of the proof.

It is intended in the first section of this paper to restate this theorem of Volterra, and to give the details of the proof. The problem may be stated somewhat more explicitly as follows. Given the equation

$$\frac{\partial u(\xi, \tau)}{\partial \tau} = f[\xi, \tau, u(\xi')], \tag{1}$$

where ξ, ξ' are real variables on the range (0, 1), τ is a real variable on $|\tau - \tau_0| \leqq \alpha$, $u(\xi')$ has the range of continuous functions for which $\max |u(\xi') - u_0(\xi')| \leqq \beta$, and $f[\xi, \tau, u]$ is a functional eliminating the argument ξ' and yielding for each ξ, τ, u of the above ranges a real number.

* See, for example, the two articles by Bliss: "The Solutions of Differential Equations of the First Order as Functions of the Initial Values," *Annals of Mathematics*, 2d Series, Vol. 6 (1905), p. 49; "Solutions of Differential Equations as Functions of the Constants of Integration," *Bulletin of the American Mathematical Society*, Vol. XXIV (1918), p. 15. For other references, see Encyclopädie der Mathematischen Wissenschaften, II, A4a, p. 195, 200.

† H. Von Koch, "Ofversigt af Konliga Vetenskaps Akademiens Fordhandligar," Vol. 56 (1899), pp. 395–411.

F. R. Moulton, *Proceedings of the National Academy of Sciences*, Vol. 1, pp. 350–354.

W. L. Hart, "Differential Equations and Implicit Functions in Infinitely Many Variables, "*Transactions of the American Mathematical Society*, Vol. 18 (1917), pp. 125–160.

‡ K. Kowalewski, "Ueber Funktionraume II," *Wiener Berichte*, Vol. 120 (1911), ab. 2a.

§ Volterra, "Equazioni integro-differenziali ed equazioni alle derivate funzionali," *Rendiconti della reale Accademia dei Lincei*, 23 serie V (1914), p. 55.

The problem is to find hypotheses which are sufficient to insure the existence of a unique solution $u(\xi, \tau)$ reducing for an arbitrary value $\bar{\tau}$ of the above range for τ to an arbitrary continuous function $\bar{u}(\xi)$ of the above range for u.

After proving the existence of a solution through a fixed initial element, one is naturally interested in knowing the character of the solutions when the latter are considered as depending not only on the variables ξ, τ but also on the initial elements. It is shown in section 3 that with suitable hypotheses, the solutions are continuous functionals of the initial elements. In order to proceed farther with these questions, it was found convenient to use the concept of "difference function" due to Bliss which is a modification of Fréchet's differential. Four lemmas which are found to be useful in the sequel are proved in section 2. In section 4 a theorem is given which tells under what circumstances the solutions will possess difference functions with regard to the initial elements. This is an analogue of a corresponding theorem in differential equations with a finite number of variables, viz., the theorem concerning differentiability with respect to the initial constants. In the last section equations involving integral power series are considered and the connection with Kowalewski's paper is brought out. In a subsequent paper an application of these results will be made in proving that there exists a solution of an equation with a continuous infinitude of variables analogous to a linear partial differential equation in a finite number of variables.

§1. Existence of a Solution Through Given Initial Elements.

Let τ and ξ denote real variables and $u(\xi')$ a real-valued continuous function on the interval $0 \leqq \xi \leqq 1$. Moreover let A be a set of elements $[\xi, \tau, u(\xi')]$ defined by the conditions

$$(A) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha, \qquad ||u(\xi') - u_0(\xi')|| \leqq \beta,$$

where the notation $||\ \ ||$ means that the maximum is taken with respect to all ξ''s of the interval $(0, 1)$ and where τ_0, u_0 stand for a particular real number and a particular real continuous function of the above ranges respectively. Finally, let $f[\xi, \tau, u(\xi')]$ be a real single-valued functional eliminating the argument ξ'. With these notations in mind, one may state and prove the following existence theorem.

THEOREM 1. *Let the functional f satisfy the hypotheses*

(H_1) *For each function u for which (ξ, τ, u) belongs to (A), $f[\xi, \tau, u]$ is continuous in $0 \leqq \xi \leqq 1$, $|\tau - \tau_0| \leqq \alpha$.*

(H_2) *There exists a number κ such that for every pair of elements (ξ, τ, u'), (ξ, τ, u'') of the set (A), the inequality*

$$|f[\xi, \tau, u'] - f[\xi, \tau, u'']| \leqq \kappa ||u' - u''||$$

holds.

Under these circumstances, $|f|$ *has an upper bound* γ *in* (A)*, and there exists one and but one function* $v(\xi, \tau)$ *such that*

(C_1) $v(\xi, \tau)$, $v_\tau(\xi, \tau)$ *are continuous and the element* $[\xi, \tau, v(\xi, \tau')]$ *is in* (A) *provided that* (ξ, τ) *is in the following set* B*:*

$$(B) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho,$$

where ρ *is the smaller of* α *and* β/γ*, and* γ *is an upper bound of* $|f[\xi, \tau, u]|$ *in the set* (A)*.*

$$(C_2) \qquad v(\xi, \tau_0) = u_0(\xi).$$

$$(C_3) \qquad v_\tau(\xi, \tau) = f[\xi, \tau, v(\xi', \tau)].$$

It will be found helpful both in simplifying the proof and as a means for future reference to consider the following lemmas.

LEMMA 1. *If the functional* f *satisfies the hypotheses of theorem* 1*, then it is bounded, i.e., there exists a number* γ *independent of whatever element* (ξ, τ, u) *of the set* (A)*, is considered for which* $|f[\xi, \tau, u]| \leqq \gamma$.

For, one may write

$$|f[\xi, \tau, u]| \leqq |f[\xi, \tau, u] - f[\xi, \tau, u_0]| + |f[\xi, \tau, u_0]|,$$

where the elements (ξ, τ, u) and (ξ, τ, u_0) are taken in the set (A). By (H_2) it is clear that the first expression on the right-hand side is less than $\kappa\beta$ and by (H_1) the last term is at most μ where $\mu = ||f[\xi, \tau, u_0]||$.

LEMMA 2. *If* $u(\xi, \tau)$ *is continuous in* (B) *and the element* $[\xi, \tau, u(\xi', \tau)]$ *is in* (A) *for every* (ξ, τ) *of* (B)*, then* $f[\xi, \tau, u]$ *is continuous in* (B)*.*

Since the element (ξ, τ, u) is in (A), one has by (H_2) the following inequality.

$$\begin{aligned} |f[\xi + \Delta\xi, \tau + \Delta\tau, u[\xi', \tau + \Delta\tau)] - f[\xi, \tau, u(\xi', \tau)]| \\ \leqq |f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau + \Delta\tau)] - f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)]| \\ + |f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)] - f[\xi, \tau, u(\xi', \tau)]| \\ \leqq \kappa\,||u(\xi', \tau + \Delta\tau) - u(\xi', \tau)|| \\ + |f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)] - f[\xi, \tau, u(\xi', \tau)]|. \end{aligned}$$

But, since $u(\xi, \tau)$ is continuous in (B), $||u(\xi', \tau + \Delta\tau) - u(\xi', \tau)||$ can be made as small as desired provided $|\Delta\tau|$ is taken small enough. Also, the expression $|f[\xi + \Delta\xi, \tau + \Delta\tau, u(\xi', \tau)] - f[\xi, \tau, u(\xi', \tau)]|$ can be made arbitrarily small by restricting sufficiently $|\Delta\xi|$, $|\Delta\tau|$, as follows from (H_1). Hence, the truth of the lemma follows.

Now, let $\{v_\nu(\xi, \tau)\}$ be a sequence of functions defined by the relations

$$(2) \qquad \begin{aligned} v_0(\xi, \tau) &= u_0(\xi), \\ v_{\nu+1}(\xi, \tau) &= u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu(\xi', \tau')]d\tau', \qquad (\nu = 0, 1, \cdots). \end{aligned}$$

LEMMA 3.

(1) *Each* $v_\nu(\xi, \tau)$ *is continuous in* (B).

(2) *If* (ξ, τ) *is in* (B), *then for each* ν *the element* $[\xi, \tau, v_\nu(\xi', \tau)]$ *is in* (A).

(3) *For each* ν *the functional* $f[\xi, \tau, v_\nu(\xi', \tau)]$ *is continuous in* (B).

By (H_1) the conclusions of the lemma including (3) are true for $\nu = 0$. Moreover since by Lemma 1, one has the inequality

$$|v_1(\xi, \tau) - u_0(\xi)| \leqq \left| \int_{\tau_0}^{\tau} f[\xi, \tau', u_0(\xi')]d\tau' \right| \leqq \gamma|\tau - \tau_0|,$$

the conclusion (2) follows for $\nu = 1$ and consequently (3) for $\nu = 1$ by Lemma 2. Thus $v_2(\xi, \tau)$ is continuous in (B). By sequentially using

$$|v_{\nu+1}(\xi, \tau) - u_0(\xi)| \leqq \gamma|\tau - \tau_0|$$

and Lemma 2, one easily obtains the results of the Lemma.

LEMMA 4. *The sequence* $\{v_\nu(\xi, \tau)\}$ *converges uniformly in* (B) *to a function* $v(\xi, \tau)$ *continuous in* (B), *and such that* (ξ, τ, v) *is in* (A) *whenever* (ξ, τ) *is in* (B).

From the defining equations (2) one has

$$|v_1(\xi, \tau) - v_0(\xi, \tau)| \leqq \mu|\tau - \tau_0|,$$

where $\mu = \| |f[\xi, \tau, u_0]| \|$ in (A). To complete the induction, assume

$$|v_\nu(\xi, \tau) - v_{\nu-1}(\xi, \tau)| \leqq \frac{\mu}{\kappa} \frac{\kappa^\nu |\tau - \tau_0|^\nu}{\nu!}.$$

It follows from this and hypothesis (H_2) that

$$|v_{\nu+1}(\xi, \tau) - v_\nu(\xi, \tau)| = \left| \int_{\tau_0}^{\tau} \{f[\xi, \tau', v_\nu(\xi', \tau')] - f[\xi, \tau', v_{\nu-1}(\xi', \tau')]\} d\tau' \right|$$

$$\leqq \left| \int_{\tau_0}^{\tau} \frac{\mu}{\kappa} \frac{\kappa^{\nu+1}}{\nu!} |\tau' - \tau_0|^\nu d\tau' \right| \leqq \frac{\mu}{\kappa} \frac{\kappa^{\nu+1} |\tau - \tau_0|^{\nu+1}}{(\nu + 1)!}.$$

Now, consider the series

(3) $v(\xi, \tau) = u_0(\xi) + [v_1(\xi, \tau) - u_0(\xi)] + [v_2(\xi, \tau) - v_1(\xi, \tau)] + \cdots$.

The absolute value of every term of (3) is less than the corresponding term of the expansion of $(e^{\kappa|\tau - \tau_0|} - 1)\mu/\kappa$. Since this dominating series converges surely for all elements (ξ, τ) of (B), it follows that the series (3) converges absolutely uniformly in (B) and hence uniformly for all (ξ, τ) of the same set. Moreover, since each term of (3) is continuous in (B) it follows that $v(\xi, \tau)$ is continuous in (B) and hence by Lemma 2, the last conclusion of Lemma 4 follows.

LEMMA 5. *The function* $v(\xi, \tau)$ *satisfies the relation*

$$v(\xi, \tau) = u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau', \qquad \text{for } \nu \geqq \nu_0.$$

For, by Lemma 4 there exists a sufficiently large index ν_0 such that the maximum with respect to both (ξ, τ) of (B) of the expression $|v_\nu(\xi, \tau) - v(\xi, \tau)|$

can be made as small as desired, say less than ϵ. Hence, since the elements (ξ, τ, v_ν) and (ξ, τ, v) are in (A), it follows by (H_2) of Theorem 1 that

$$|\int_{\tau_0}^{\tau}\{f[\xi, \tau', v_\nu(\xi', \tau')] - f[\xi, \tau', v(\xi', \tau')]\}d\tau' \leqq \kappa\epsilon|\tau - \tau_0|$$

or

$$\lim_{\nu\to\infty} \int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu(\xi', \tau')]d\tau' = \int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau'.$$

But, by Lemma 4, $v(\xi, \tau) = \lim_{\nu\to\infty} v_\nu(\xi, \tau)$ so that it follows from the relations (2) that

$$v(\xi, \tau) = u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v(\xi', \tau')]d\tau!$$

as desired.

On differentiating the last relation with respect to τ, one sees that $v_\tau(\xi, \tau)$ is continuous in (B) and satisfies the differential equation (1). Moreover by putting $\tau = \tau_0$ in this same equation, one finds that the solution reduces to the specified initial function $u_0(\xi)$.

It remains only to prove the uniqueness of the solution.

LEMMA 6. *The equation* (1) *has only one solution* $v(\xi, \tau)$ *which is continuous in* (B), *for which* $v_\tau(\xi, \tau)$ *is continuous in* (B) *and which reduces to* $u_0(\xi)$ for $\tau = \tau_0$.

Suppose that besides $v(\xi, \tau)$ there could be another solution $\bar{v}(\xi, \tau)$ having the properties stated in the lemma. Then this solution also would have to satisfy a relation like that in Lemma 5. It would then follow by (H_2) of Theorem 1, since both (ξ, τ, v) and $(\xi, \tau, \bar{v})$ are in (A), that

$$(4) \qquad |v(\xi, \tau) - \bar{v}(\xi, \tau)| = |\int_{\tau_0}^{\tau}\{f[\xi, \tau', v(\xi', \tau')] - f[\xi, \tau', \bar{v}(\xi', \tau')]\}d\tau'| \leqq \tau\kappa\beta|\tau - \tau_0|.$$

If now (H_2) is sequentially applied to the second member of (4) and use is made of the last inequality, one obtains finally the inequality

$$|v(\xi, \tau) - \bar{v}(\xi, \tau)| \leqq \frac{2\beta\kappa^\nu|\tau - \tau_0|^\nu}{\nu!}$$

from which follows the statement of the Lemma by a passage to the limit.

This completes the proof of Theorem I.

The following theorem, which is really a Corollary of Theorem I, will be found useful.

THEOREM II. *Let a set* (A') *be defined by*

$$(A') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha, \qquad ||u - u_0|| \leqq \infty.$$

If the hypotheses of Theorem I are satisfied in this set and if $||f[\xi, \tau, u_0]||$, *then the conclusions of Theorem I will hold for the set of points* (B'),

$$(B') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \alpha.$$

Part (2) of Lemma 3 which deals with proving that the element (ξ, τ, v) is in (A') is of course unnecessary here. The smaller of α and β/γ should in this case be replaced evidently by the number α. The convergence and uniqueness proofs go through precisely as before.

§ 2. Some Definitions and Lemmas Concerning Functionals and Difference Functions.

Before proceeding to a discussion of other properties of the solutions of equation (I), it will be convenient to give a few definitions and lemmas to which frequent reference will be made in the sequel.

Let η stand for the set of elements $(\eta_1 \cdots \eta_r)$ where the $\eta_1 \cdots \eta_r$ are real variables with ranges which are composed of continuous intervals or discrete sets. For example, η_1 might have the range 1, 1/2, 1/3, $\cdots$ and all the other η's vary over (0, 1). Moreover, let u stand for the set of elements $(u_1 \cdots u_s)$ where $u_1 \cdots u_s$ are real continuous functions of ξ' on the interval $0 \leqq \xi' \leqq 1$. Consider now a real-valued functional operation $v[\xi, \eta, u] = v[\xi, \eta_1 \cdots \eta_r, u_1 \cdots u_s)$ defined by the relations

$$(R) \qquad \begin{aligned} &0 \leqq \xi \leqq 1, \qquad |\eta_i - \eta_i^{(0)}| \leqq \lambda_i \qquad (i = 1 \cdots r), \\ &||u_i - u_i^{(0)}|| \leqq \mu_i \qquad (i = 1 \cdots s). \end{aligned}$$

In other words, $v[\xi, \eta, u]$ is of such a character that whenever the element (ξ, η, u) of the set (R) is given, the correspondence v determines a real number. Unless otherwise specified, attention will be confined to η's which represent sets of variables having continuous ranges.

Definition.—$v[\xi, \eta, u]$ is said to be continuous in the set (R) if for every positive number ϵ and for every u for which (ξ, η, u) is in (R), there exists another positive number $\delta_{\epsilon,u}$ which is independent of (ξ, η) such that the inequality $|v[\bar{\xi}, \bar{\eta}, \bar{u}] - v[\xi, \eta, u]| \leqq \epsilon$ holds if (ξ, η, u) and $(\xi, \bar{\eta}, \bar{u})$ are in (R) and if $|\bar{\xi} - \xi| \leqq \delta_{\epsilon,u}$, $|\bar{\eta} - \eta| \leqq \delta_{\epsilon,u}$, $||\bar{u} - u|| \leqq \delta_{\epsilon,u}$.

The notation $|\bar{\eta} - \eta|$ means of course $|\bar{\eta}_i - \eta_i|$, $(i = 1 \cdots r)$ and $||\bar{u} - u||$ stands for $||\bar{u}_i - u_i||$, $(i = 1 \cdots s)$.

LEMMA 1. *If*

(1) *$v_\nu[\xi, \eta, u]$ is continuous in (R),*

(2) *The sequence $\{v_\nu[\xi, \eta, u]\}$ tends to a limit $v[\xi, \eta, u]$ uniformly with respect to all elements $[\xi, \eta, u]$ of (R), then, $v[\xi, \eta, u]$ is continuous in (R).*

In the inequality

$$\begin{aligned} |v[\bar{\xi}, \bar{\eta}, \bar{u}] - v[\xi, \eta, u]| \leqq\ & |v[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}]| \\ & + |v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\xi, \eta, u]| + |v_{\nu_0}[\xi, \eta, u] - v[\xi, \eta, u]| \end{aligned}$$

one can choose ν_0 so large that

$$|v[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}]| + |v_{\nu_0}[\xi, \eta, u] - v[\xi, \eta, u]| \leqq \frac{2\epsilon}{3}$$

by the uniformity of the convergence. Moreover by the continuity of $v_{\nu_0}[\xi, \eta, u]$ there exists a number $\delta_{\epsilon, u, \nu_0}$ for which

$$|v_{\nu_0}[\bar{\xi}, \bar{\eta}, \bar{u}] - v_{\nu_0}[\xi, \eta, u]| \leqq \epsilon/3.$$

Take this $\delta_{\epsilon, u, \nu_0}$ as the δ associated with the continuity of $v[\xi, \eta, u]$.

Let $(\bar{R})$ be a set of elements $[\bar{\eta}, \bar{u}]$ where $\bar{\eta}$ stands for a set $(\bar{\eta}_1 \cdots \bar{\eta}_r)$ with each $\bar{\eta}_i$ ranging over the whole linear continuum and $\bar{u}$ stands for the set of functions $(\bar{u}_1 \cdots \bar{u}_s)$ where the $\bar{u}_i$ are arbitrary continuous functions of ξ' on the interval (0, 1). In all that follows in this section the elements (ξ, η, u) and $(\bar{\xi}, \bar{\eta}, \bar{u})$ shall be understood as belonging to (R) and $(\bar{\eta}, \bar{u})$ as belonging to $(\bar{R})$.

Definition of the function $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$.

(1) *a is continuous in* $[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *in the sense defined above.*

(2) $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] = \gamma_1 a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2]$, *where* $(\tilde{\eta}_1, \tilde{u}_1)$, $(\tilde{\eta}_2, \tilde{u}_2)$ *are elements of* $(\bar{R})$ *and* γ_1, γ_2 *are arbitrary real numbers.*

(3) *There exists a positive number* μ *independent of* $[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *for which*

$$|a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]| \leqq \mu ||\tilde{\eta}, \tilde{u}||.$$

The notation $||\tilde{\eta}, \tilde{u}||$ means the larger of $[||\tilde{\eta}||, ||\tilde{u}||]$.

Definition of a difference function.

$a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *is said to be a difference function of* $f[\xi, \eta, u]$ *if* a *has the properties* (1), (2), (3) *just defined and satisfies the relation*

$$f[\bar{\xi}, \bar{\eta}, \bar{u}] - f[\xi, \eta, u] = a[\xi, \eta, u, \bar{\eta}, \bar{u}; \bar{\eta} - \eta, \bar{u} - u].$$

Besides the properties (1), (2), (3) of a there will be occasion to consider a fourth property.

(1′) *If in* a, *one substitutes for* u, $\bar{u}$, $\tilde{u}$ *the functionals* $v'[\xi, \eta', u']$, $v''[\xi, \eta'', u'']$, $v'''[\xi, \eta''', u''']$ *respectively, where* v', v'', v''' *are continuous in the sense above defined, then the resulting expression considered as depending on* $\xi, \eta, \bar{\eta}, \tilde{\eta}, \eta', \eta'', \eta''', u', u'', u'''$ *is also continuous in the same sense.*

It should be understood in (1′) that η', η'', η''' may denote sets of variables which have both discrete and continuous ranges. If for example η_1 has the range 1, 1/2, 1/3 $\cdots$, then $v(\eta_1)$ is continuous at the point 0 if $|v(\eta_1') - v(0)| \leqq \epsilon$ whenever $|\eta_1'| \leqq \delta$ or what amounts to the same thing $\lim v(\eta_1) = v(0)$ when $\lim \eta_1 = 0$. The u', u'', u''' must have ranges given by (R). This last property is denoted by (1′) because it is similar to (1) but much stronger. It will be understood therefore that whenever (1′) is used it will replace (1).

The following lemmas will be found very useful.

LEMMA 2. *If* $f[\xi, \eta, u]$ *has a difference function* $a[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ *in*

the set (R), it satisfies the Lipschitz condition described in hypothesis (H_2) of Theorem I in this set.

For, taking $\bar{\eta} = \eta$ and u, $\bar{u}$ as single real continuous functions, one sees from the definition of a difference function and property (3) that

$$|f[\xi, \eta, \bar{u}] - f[\xi, \eta, u]| \leqq \mu||\bar{u} - u||.$$

LEMMA 3. *If $f[\xi, \eta, u]$ has a difference function in (R), it is continuous in (R) in the sense described above.*

For, if in the inequality

$$|f[\bar{\xi}, \bar{\eta}, \bar{u}] - f[\xi, \eta, u]| \leqq |f[\bar{\xi}, \bar{\eta}, \bar{u}] - f[\bar{\xi}, \eta, u]| + |f[\bar{\xi}, \eta, u] - f[\xi, \eta, u]|$$

one takes $||\bar{\eta} - \eta||$, $||\bar{u} - u||$ sufficiently small, one can make $|f[\xi, \bar{\eta}, \bar{u}] - f[\xi, \eta, u]|$ as small as desired, since it is less than $\mu||\bar{\eta} - \eta, \bar{u} - u||$. Furthermore by property (1) of the difference function the second expression on the right tends to zero with $|\bar{\xi} - \xi|$. It is to be noted here that because of the character of the μ the continuity is surely of the type defined in the beginning of this section and even stronger.

LEMMA 4. *Let $v_\nu[\xi, \eta, u]$ be a sequence of functionals with the properties:*

(1) *The sequence converges to a limit function $v[\xi, \eta, u]$.*

(2) *Each $v_\nu[\xi, \eta, u]$ has a difference function $b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ with all of which are associated the same constant μ.*

(3) *The sequence $\{b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]\}$ converges to a limit $b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ uniformly with respect to all (ξ, η, u) in (R), $(\bar{\xi}, \bar{\eta}, \bar{u})$ in (R) and $(\tilde{\eta}, \tilde{u})$ for which $||\tilde{\eta}, \tilde{u}|| \leqq 1$.*

Then, the limit $b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ has the properties (1), (2) and (3) and is a difference function for $v[\xi, \eta, u]$.

From (2) one has the relation

$$v_\nu[\xi, \bar{\eta}, \bar{u}] - v_\nu[\xi, \eta, u] = b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \eta - \eta, \bar{u} - u], \qquad \nu = 1, 2, \cdots,$$

so that passing to the limit as is allowable by (1) and (3), it is clear that

$$v[\xi, \bar{\eta}, \bar{u}] - v[\xi, \eta, u] = b[\xi, \eta, u, \eta, \bar{u}, \bar{\eta} - \eta, \bar{u} - u].$$

It remains to show that $b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}, \tilde{u}]$ has the properties (1), (2), (3) of a difference function. By (2) and (3) one sees that

$$\begin{aligned} &b[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] \\ &\quad = \lim_{\nu \to \infty} b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] \\ &\quad = \lim_{\nu \to \infty} \{\gamma_1 b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 b_\nu[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2]. \end{aligned}$$

Since each term on the right-hand side separately converges, it follows that

$$\begin{aligned} &b[\xi, \eta, u, \eta, \bar{u}; \gamma_1\tilde{\eta}_1 + \gamma_2\tilde{\eta}_2, \gamma_1\tilde{u}_1 + \gamma_2\tilde{u}_2] \\ &\qquad = \gamma_1 b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_1, \tilde{u}_1] + \gamma_2 b[\xi, \eta, u, \bar{\eta}, \bar{u}; \tilde{\eta}_2, \tilde{u}_2] \end{aligned}$$

which proves property (2). Moreover since each b_ν is continuous in $\xi, \eta, u, \bar{\eta}, \bar{u}, \tilde{\eta}, \tilde{u}$ and since the sequence $\{b_\nu\}$ converges uniformly to b, it follows from Lemma I that the limit functional b will also be continuous in the same arguments. Finally, property (3) is an immediate consequence of hypothesis (2).

§ 3. CONTINUITY OF THE SOLUTIONS WITH RESPECT TO THE INITIAL ELEMENTS.

Consider now a set (A_0) of elements (ξ, τ, u) defined by the inequalities

$$(A_0) \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_{00}| \leqq \alpha + \delta, \qquad \| u - u_{00}\| \leqq \beta + \delta,$$

where (ξ, τ_{00}, u_{00}) is a particular element. Assume that all the hypotheses of Theorem I are satisfied in this set. Consider also a set of elements (ξ, τ, τ_0, u_0) given by the relations,

$$(B_0) \quad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho, \qquad |\tau_0 - \tau_{00}| \leqq \delta, \qquad \| u_0 - u_{00}\| \leqq \delta,$$

where ρ as before is the smaller of α and β/γ.

The set (A) associated with every element (τ_0, u_0) for which $|\tau_0 - \tau_{00}| \leqq \delta$ and $\| u_0 - u_{00}\| \leqq \delta$ is a part of (A_0) as is clear from the inequalities

$$|\tau - \tau_{00}| \leqq |\tau - \tau_0| + |\tau_0 - \tau_{00}| \leqq \alpha + \delta,$$
$$\| u - u_{00} \| \leqq \| u - u_0 \| + \| u_0 - u_{00} \| \leqq \beta + \delta.$$

From this it follows that if one defines a sequence of approximating functionals $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ by equations (2) where now $[\xi, \tau, \tau_0, u_0]$ are all thought of as varying in the set (B_0), one can prove precisely as in Lemma 3, section 1, that these functionals are all defined in (B_0) and that the sequence converges uniformly in (B_0) to a solution $v[\xi, \tau, \tau_0, u_0]$ of the differential equation (1). One has thus secured a set of solutions which for all possible variations of the initial element $[\xi, \tau, \tau_0, u_0]$ in (B) has a common interval of definition $|\tau - \tau_0| \leqq \rho$.

THEOREM III. *If the hypotheses of Theorem I are satisfied in the set (A_0), then in the set (B_0) the solutions $v[\xi, \tau, \tau_0, u_0]$ are continuous functionals of their arguments in the sense of section 2.*

In the notations of section 2 the conclusion of this theorem has reference to a set $(R) = (B_0)$ with the elements $\xi = \xi$, $\eta = (\tau, \tau_0)$ and $u = u_0$. The proof of the theorem can be made to rest on the following

LEMMA. *If*

(1) *$f[\xi, \tau, u]$ satisfies the hypotheses of Theorem I in the set (A_0),*

(2) *$v[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0),*

then, the functionals $g[\xi, \tau, \tau_0, u_0] = f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]$ and $\int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]d\tau'$ are also continuous in (B_0) in the same sense.

Consider any two elements $[\xi, \tau, \tau_0, u_0]$, $[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ of (B_0). One may write

$$(5)\quad \begin{aligned} |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0]| \leqq & |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0]| \\ & + |g[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0] - g[\xi, \tau, \tau_0, u_0]|. \end{aligned}$$

Since f satisfies the Lipschitz condition in (A_0), it follows that the first expression on the right will not exceed $\kappa|v[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, u_0]|$. From the continuity of v it follows that this may be made as small as desired if $(\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0)$ is taken sufficiently near (ξ, τ, τ_0, u_0). It is to be remarked moreover that the associated δ is independent of ξ, τ, τ_0. Also, the second expression on the right-hand side of (5) can be made as small as one pleases, since it is equal to $|f[\bar{\xi}, \bar{\tau}, v[\xi', \bar{\tau}, \tau_0, u_0]] - f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]|$ and hence for each u_0 is uniformly continuous in ξ, τ, τ_0. This proves the first part of the Lemma.

Now, set

$$h[\xi, \tau, \tau_0, u_0] = \int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]d\tau'.$$

Then, it is clear that

$$\begin{aligned} |h[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]| \leqq & \left|\int_{\tau}^{\bar{\tau}} g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0]d\tau'\right| \\ & + \left|\int_{\tau_0}^{\tau_0} g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0]d\tau'\right| \\ & + \left|\int_{\tau_0}^{\tau} \{g[\bar{\xi}, \tau', \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau', \tau_0, u_0]\}d\tau'\right|. \end{aligned}$$

Hence, applying the first part of the Lemma and Lemma 1 of section 1 to the first two terms on right, and the fact that $g[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0), one sees that

$$|h[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0]| \leqq \gamma(|\bar{\tau} - \tau| + |\bar{\tau}_0 - \tau_0| + \epsilon|\tau - \tau_0|)$$

if $(\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0)$ is sufficiently near to (ξ, τ, τ_0, u_0) which proves that $h[\xi, \tau, \tau_0, u_0]$ is continuous in (B_0). It is important to note that the uniformity of the continuity of $g[\xi, \tau, \tau_0, u_0]$ with respect to τ was needed in this discussion.

Consider now the approximating functionals

$$(2')\quad \begin{aligned} v_0[\xi, \tau, \tau_0, u_0] &= u_0(\xi), \\ v_{\nu+1}[\xi, \tau, \tau_0, u_0] &= u_0(\xi) + \int_{\tau_0}^{\tau} f[\xi, \tau', v_\nu[\xi', \tau', \tau_0, u_0]]d\tau', \\ & \qquad \nu = 0, 1, 2, \cdots. \end{aligned}$$

Suppose $[\xi, \tau, \tau_0, u_0]$ is any element of (B_0). Then for $\nu = 0$ one has

$$|\Delta v_0| = |v_0[\bar{\xi}, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v_0[\xi, \tau, \tau_0, u_0]| = |\bar{u}_0(\bar{\xi}) - u_0(\xi)|$$

so that

$$|\Delta v_0| \leqq |\bar{u}_0(\bar{\xi}) - u_0(\bar{\xi})| + |u_0(\bar{\xi}) - u_0(\xi)|.$$

But, if ϵ is arbitrarily assigned there exists a δ_{ϵ, u_0} independent of ξ such that if $|\bar{\xi} - \xi| \leqq \delta_{\epsilon, u_0}$ then $|u_0(\bar{\xi}) - u_0(\xi)| \leqq \epsilon$. This δ is independent of ξ

since u_0 is a continuous function on a closed interval. Also there exists a δ_ϵ such that if $\| \bar{u}_0 - u_0 \| \leqq \delta_\epsilon$ then $| \bar{u}_0(\bar{\xi}) - u_0(\bar{\xi}) | \leqq \epsilon$. This shows that $v_0[\xi, \tau, \tau_0, u_0]$ possesses precisely that type of continuity heretofore considered and may therefore be used as the $v[\xi, \tau, \tau_0, u_0]$ of the preceding Lemma. Thus by successive applications of this Lemma one gets the result that all the $v_\nu[\xi, \tau, \tau_0, u_0]$ are continuous in (B_0). Furthermore, as has already been remarked at the beginning of this section, the sequence $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ converges to a solution $v[\xi, \tau, \tau_0, u_0]$ uniformly in (B_0). Hence, using Lemma I of section 2 with $\eta = (\tau, \tau_0)$, $u = u_0$ one obtains the conclusion of Theorem III.

§ 4. Differentiability of the Solutions with Respect to the Initial Elements.

The main result of this section is embodied in the following

THEOREM IV. *If*

(H_1) *The hypotheses of Theorem I are satisfied in* (A_0),

(H_2) $f[\xi, \tau, u]$ *has a difference function* $a[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau}, \bar{u}]$ *in* (A_0),

$$f[\xi, \bar{\tau}, \bar{u}] - f[\xi, \tau, u] = a[\xi, \tau, u, \bar{\tau}, \bar{u}; \bar{\tau} - \tau, \bar{u} - u],$$

(H_3) *The difference function of* f *has the additional property* $(1')$ *of section 2 in* (A_0),

then, the solution $v[\xi, \tau, \tau_0, u_0]$ *has in* (B_0) *a difference function* $b[\xi, \tau, \tau_0, u_0\bar{\tau}, \bar{\tau}_0\bar{u}_0; \bar{\tau}\bar{\tau}_0\bar{u}_0]$.

The style of proof will be to interpret the sequence $\{v_\nu[\xi, \tau, \tau_0, u_0]\}$ with $\eta = (\tau, \tau_0)$ and $u = u_0$ of Lemma 4, section 2, as the sequence of approximating functionals $(2')$. If then one can show that all the hypotheses of that Lemma are satisfied here, Theorem IV will be proved.

LEMMA 1. *If*

(1) $f[\xi, \tau, u]$ *satisfies the hypotheses of Theorem IV,*

(2) $v[\xi, \tau, \tau_0, u_0]$ *has a difference function* $b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0\bar{u}_0; \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ *in* (B_0),

(3) $(\xi, \tau, v[\xi', \tau, \tau_0, u_0])$ *is in* (A_0) *for every* (ξ, τ, τ_0, u_0) *of* (B_0),

then, $g[\xi, \tau, \tau_0, u_0] = f[\xi, \tau, v[\xi', \tau, \tau_0, u_0]]$ *and* $\int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]d\tau'$ *also have difference functions in* (B_0).

By hypothesis (1)

$$g[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0] = a[\xi, \tau, v, \bar{\tau}, \bar{v}; \bar{\tau} - \tau, \bar{v} - v],$$

where $v = v[\xi, \tau, \tau_0, u_0]$ and $\bar{v} = v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$. Moreover

$$v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v[\xi, \tau, \tau_0, u_0] = b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0].$$

Hence,

$$g[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - g[\xi, \tau, \tau_0, u_0] = c[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0],$$

where

$$c[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0 \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] = a[\xi, \tau, v, \bar{\tau}, \bar{v}; \tilde{\tau}, b[\xi', \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0]].$$

It remains to show that c has properties (1), (2) and (3) of a difference function. Property (1) of c is an immediate consequence of the property (1′) possessed by the difference function a. Property (2) follows from property (2) of b and a. Since (ξ, τ, v), $(\xi, \bar{\tau}, \bar{v})$ are in (A_0), it is clear from property (3) of a that $|c| \leqq \mu \,||\, \tilde{\tau}, b \,||$ and hence from property (3) of b that $|c| \leqq \mu\lambda \,||\, \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0 \,||$, where λ is the constant associated with b. Thus property (3) of c is proved. This completes the proof that $g[\xi, \tau, \tau_0, u_0]$ has a difference function in (B).

Consider now the functional $h[\xi, \tau, \tau_0, u_0] = \int_{\tau_0}^{\tau} g[\xi, \tau', \tau_0, u_0]$. By means of an easy computation it is found that

$$\begin{aligned} k[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] &= \tilde{\tau} \int_0^1 g[\xi, \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0] d\theta \\ &+ \tilde{\tau}_0 \int_1^0 g[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0] d\theta \\ &+ \int_{\tau_0}^{\tau} c[\xi, \tau', \tau_0, u_0, \tau', \bar{\tau}_0, \bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0] d\tau', \end{aligned} \tag{6}$$

where

$$k[\xi, \tau, \tau_0, u_0 \overline{\tau\tau_0} \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0] = h[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - h[\xi, \tau, \tau_0, u_0].$$

Properties (2) and (3) of k follow readily from the identity (6), properties (2) and (3) of g, the continuity of c, and elementary properties of integrals. The proof of property (1) follows readily from the fact that g and c are continuous in the sense of section 2.

LEMMA 2. *Every approximating functional $v_\nu[\xi, \tau, \tau_0, u_0]$ has a difference function b_ν in (B_0) if f satisfies the hypotheses of Theorem IV.*

It is clear from (2′) that $v_0[\xi, \tau, \tau_0, u_0]$ has a difference function $b_0 = \tilde{u}_0$. Then from Lemma 1, it will follow that every v_ν has a difference function b_ν where

$$b_\nu[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau} - \tau, \bar{\tau}_0 - \tau_0, \bar{u}_0 - u_0] = v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] - v_\nu[\xi, \tau, \tau_0, u_0], \qquad \nu = 1, 2, \cdots.$$

It can be shown from the defining equations (2′) that

$$\begin{aligned} b_{\nu+1}[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \tilde{\tau}, \tilde{\tau}_0, \tilde{u}_0] &= \tilde{u}_0(\xi) \\ &+ \tilde{\tau} \int_0^1 f[\xi, \tau + \theta(\bar{\tau} - \tau), v_\nu[\xi', \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0] d\theta \\ &+ \tilde{\tau}_0 \int_1^0 f[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), v_\nu[\xi', \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0] d\theta \\ &+ \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, b_\nu[\xi', \tau', \tau_0, u_0, \tau' \bar{\tau}_0, \bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0] d\tau' \end{aligned} \tag{7}$$

where v_ν and $\bar{v}_\nu$ stand for $v_\nu[\xi, \tau, \tau_0, u_0]$ and $v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ respectively.

LEMMA 3. *The difference functions b_ν of the approximating functionals have a common constant λ associated with them.*

In the first place it is clear from the definition of b_0 that

$$|b_0| \leqq \|\,|\tilde{\tau}|,\ |\tilde{\tau}_0|,\ |\tilde{u}_0|\,\|.$$

Furthermore it follows from (7) and the properties of f and a that if there exist functions $b_\nu(\tau)$ such that

$$|b_\nu| \leqq \beta_\nu(\tau)\,\{\|\,|\tilde{\tau}|,\ |\tilde{\tau}_0|,\ |\tilde{u}_0|\,\|\},$$

then

$$|b_{\nu+1}| \leqq \{1 + 2\gamma + |\textstyle\int_{\tau_0}^{\tau} \kappa\beta_\nu(\tau')d\tau'|\}\,\{\|\,|\tilde{\tau}|,\ |\tilde{\tau}_0|,\ |\tilde{u}_0|\,\|\},$$

so that

$$\beta_{\nu+1}(\tau) = 1 + 2\gamma + |\textstyle\int_{\tau_0}^{\tau} \kappa\beta_\nu(\tau')d\tau'|.$$

One obtains therefore the following expressions of the functions $\beta_\nu(\tau)$

$$\beta_0 = 1,$$

$$\beta_1 = (1 + 2\gamma) + \frac{\kappa|\tau - \tau_0|}{1!},$$

$$\beta_2 = (1 + 2\gamma) + \frac{(1 + 2\gamma)^{\kappa|\tau - \tau_0|}}{1!} + \frac{\kappa^2|\tau - \tau_0|^2}{2!} \leqq (1 + 2\gamma)e^{\kappa\rho},$$

$$\vdots$$

$$\beta_{\nu+1} = (1 + 2\gamma)\left\{1 + \frac{\kappa|\tau - \tau_0|}{1!} + \cdots \frac{\kappa^\nu|\tau - \tau_0|^\nu}{\nu!}\right\} + \frac{\kappa^{\nu+1}|\tau - \tau_0|^{\nu+1}}{(\nu + 1)!} \leqq (1 + 2\gamma)e^{\kappa\rho}.$$

Hence the common constant λ is given by $(1 + 2\gamma)e^{\kappa\rho}$ where $\rho = |\tau - \tau_0|$.

LEMMA 4. *If $p_\nu[\xi, \tau, \tau_0, u_0]$ and $p[\xi, \tau, \tau_0, u_0]$ are continuous in (B_0) and $\lim p_\nu = p$ uniformly in (B_0), then $\lim c_\nu(p_\nu) = c(p)$, where*

$$c_\nu[p_\nu] = \textstyle\int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p_\nu[\xi, \tau, \tau_0, u_0]]d\tau',$$
$$c[p] = \textstyle\int_{\tau_0}^{\tau} a[\xi, \tau', v, \tau', \bar{v}; 0, p[\xi, \tau, \tau_0, u_0]]d\tau'.$$

It is seen that $c[p]$ is obtained from $c_\nu[p_\nu]$ by substituting for v_ν and $\bar{v}_\nu$ the limits v and $\bar{v}$ respectively as $\nu \to \infty$, and putting for p_ν the limit p. One has from the linearity property of

$$c[p] - c_\nu[p_\nu] = \textstyle\int_{\tau_0}^{\tau} \{a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p] - a[\xi, \tau', v, \tau', \bar{v}; 0, p]\}d\tau' + \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, p_\nu - p].$$

The integrand of the second term on the right does not exceed $\mu \,\|\, p_\nu - p \,\|$ so that it can be made as small as desired because of the uniformity of the

approach of p_ν to p. In the first term regard v_ν and $\bar{v}_\nu$ as depending upon $\nu', \xi, \tau, \tau_0, u_0$ and $\nu', \xi, \bar{\tau}, \bar{\tau}_0, u_0$, where $\nu' = 1/\nu$. Since $\lim_{\nu\to\infty} v_\nu[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0] = v[\xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ and $\lim_{\nu\to\infty} v_\nu[\xi, \tau, \tau_0, u_0] = v[\xi, \tau, \tau_0, u_0]$ uniformly in (B_0), it follows that $v[\nu', \xi, \tau, \tau_0, u_0]$ and $v[\nu', \xi, \bar{\tau}, \bar{\tau}_0, \bar{u}_0]$ are continuous functions of their arguments in the sense of section 2 so that property (1′) of a difference function is applicable, thus showing that $\lim_{\nu\to\infty} c_\nu[p_\nu] = c[p]$ as desired.

LEMMA 5. *The sequence* $\{b_\nu\}$ *converges uniformly in* (B_0) *to a limit* b *given by the expression* (11) *below.*

By Lemma 2 of this section one may write

$$b_{\nu+1} = d_\nu + c_\nu[b_\nu], \tag{8}$$

where

$$d_\nu = \tilde{u}_0(\xi) + \tilde{\tau}\int_0^1 f[\xi, \tau + \theta(\bar{\tau} - \tau), v_\nu[\xi', \tau + \theta(\bar{\tau} - \tau), \bar{\tau}_0, \bar{u}_0]]d\theta; \\ + \tilde{\tau}_0\int_1^0 f[\xi, \tau_0 + \theta(\bar{\tau}_0 - \tau_0), v_\nu[\xi', \tau_0 + \theta(\bar{\tau}_0 - \tau_0), \bar{\tau}_0, \bar{u}_0]]d\theta \tag{9}$$

and

$$c_\nu[b_\nu] = \int_{\tau_0}^{\tau} a[\xi, \tau', v_\nu, \tau', \bar{v}_\nu; 0, b_\nu[\xi', \tau', \tau_0, u_0, \tau', \bar{\tau}_0\bar{u}_0; 0, \tilde{\tau}_0, \tilde{u}_0]]d\tau'. \tag{10}$$

Repeating (8) ν times one obtains

$$b_{\nu+1} = d_\nu + c_\nu[d_{\nu-1}] + c_\nu c_{\nu-1}[d_{\nu-2}] + \cdots + c_\nu c_{\nu-1}\cdots c_1[b_0],$$

where the notation explains itself.

Consider also the infinite sum

$$b = d + c[d] + c^2[d] + \cdots + c^\kappa[d] + \cdots, \tag{11}$$

where d and $c[d]$ have precisely the form of d_ν and $c_\nu[d_\nu]$ only with v, $\bar{v}$ substituted for v_ν and $\bar{v}_\nu$ respectively, and where c^κ means that the operation c is repeated κ times. It has already been seen that each term of (10) is dominated by the corresponding term of $(1 + 2\gamma)e^{\kappa\varphi}|||\tilde{\tau}|, |\tilde{\tau}_0|, |u_0|||$. In a similar way it can be shown that the series for the last expression also dominates term by term the series (11). Hence there exists an integer κ_0 of such a character that all the terms of $b_{\nu+1}$ after the κ_0th have a sum which is in absolute value at most $\epsilon/3$ where ϵ is an arbitrarily assigned positive number. The same is true for the series representing b. Moreover, by repeated application of Lemma 4, one sees that there exists an index ν_0 such that if $\nu > \nu_0$, the first κ_0 terms of $b_{\nu+1}$ can be made to differ from the first κ_0 terms of b by a number which in absolute value is at most $\epsilon/3$. Hence $\lim b_\nu = b$ uniformly in (B_0).

Recalling now that the v_ν converge to the solution v one sees that the hypothesis $H_{(1)}$ of Lemma 4, section 2 is satisfied. Furthermore, hypothesis $H_{(2)}$ of that Lemma is contained in Lemmas 2 and 3 of this section and $H_{(3)}$ is nothing but Lemma 5. Thus the conclusions of Lemma 4 are applicable

so that the solution $v[\xi, \tau, \tau_0, u_0]$ has a difference function b in (B_0) as desired.

§ 5. Differential Equations Involving Schmidt Integral Power Series.

Let $u(\xi)$ be a real continuous function on the range (01). Then the function of $r+1$ arguments

$$u^p(\xi)u^{p_1}(\xi_1)\cdots u^{p_r}(\xi_r) \tag{12}$$

where $\xi_1 \cdots \xi_r$ also range over (01) and $p,\ p_1, \cdots p_r$ are equal or distinct integers is evidently continuous. If now (12) is multiplied by an arbitrary function $\kappa(\xi, \xi_1, \cdots \xi_r)$ continuous in its arguments and the whole expression is integrated with respect to $\xi_1 \cdots \xi_r$, then the resulting function

$$w[\xi, u] = \int_0^1 \cdots \int_0^1 \kappa(\xi, \xi_1, \cdots \xi_r)u^p(\xi)u^{p_1}(\xi_1)\cdots u^{p_r}(\xi_r)d\xi_1 \cdots d\xi_r^* \tag{13}$$

is again a continuous function of ξ. The expression

$$\mathfrak{A}[\xi, u] = w_1[\xi, u] + w_2[\xi, u] + \cdots + w_n[\xi, u]$$

is called a homogeneous integral power form of the mth order (Schmidt) if $w_1 \cdots w_n$ all have the form (13), and if for each w_i, $p + p_1 + \cdots + p_r = m$, where we may suppose always $m \leqq p$. For example, an integral power form of the 2d order has the following form,

$$\int_0^1 \cdots \int_0^1 \{\kappa_{200}u^2(\xi) + \kappa_{110}u(\xi)u(\xi_1) + \kappa_{020}u^2(\xi_1) + \kappa_{011}u(\xi_1)u(\xi_2)\}d\xi_1 \cdots d\xi_r,$$

when all the κ's are arbitrary continuous functions of the $r+1$ arguments $\xi, \xi_1, \cdots \xi_r$. This evidently reduces to

$$\alpha(\xi)u^2(\xi) + u(\xi)\int_0^1 \beta(\xi, \xi_1)u(\xi_1)d\xi_1 + \int_0^1 \gamma(\xi, \xi_1)u^2(\xi_1)d\xi_1 + \int_0^1 \int_0^1 \epsilon(\xi, \xi_1, \xi_2)u(\xi_1)u(\xi_2)d\xi_1 d\xi_2.$$

The κ's or the $\alpha, \beta, \gamma \cdots$ are called the coefficients by the form.

Let $\mathfrak{A}[\xi, u]$ be an integral power form of the mth order and let u be replaced by $u + \bar{u}$, where $\bar{u}$ is a continuous function of ξ. Then the sum of the terms involving the first power of $\bar{u}$ is called the *first differential of* $\mathfrak{A}[\xi, u]$ and is denoted by $\mathfrak{A}[\xi, u, \bar{u}]$ (Kowalewski). It is clear that when u is fixed $\mathfrak{A}[\xi, u, \bar{u}]$ is an integral power form of the 1st order in $\bar{u}$ and for $\bar{u}$ fixed, of the $(m-1)$st order in u. This differential is precisely the difference function defined in section 2 when $\bar{u} = u$. For example to compute the difference function of an integral power form of the 2d order one considers the expression

$$\begin{aligned}\mathfrak{A}[\xi, \bar{u}] - \mathfrak{A}[\xi, u] = {} & \alpha(\xi)[\bar{u}^2(\xi) - u^2(\xi)] + \bar{u}(\xi)\int_0^1 \beta(\xi, \xi_1)\bar{u}(\xi_1)d\xi_1 \\ & - u(\xi)\int_0^1 \beta(\xi, \xi_1)u(\xi_1)d\xi_1 + \int_0^1 \gamma(\xi, \xi_1)[\bar{u}^2(\xi_1) - u^2(\xi_1)]d\xi_1 \\ & + \int_0^1 \int_0^1 \epsilon(\xi, \xi_1, \xi_2)[\bar{u}(\xi_1)\bar{u}(\xi_2) - u(\xi_1)u(\xi_2)]d\xi_1 d\xi_2,\end{aligned}$$

*It is clear that the variables of integration $\xi_1, \cdots \xi_r$ may be so arranged that $p_1 \geqq p_2 \geqq \cdots \geqq p_r$ and this will be assumed throughout the section. For an account of this work see Kowalewski, loc. cit.

so that the difference function is

$$a[\xi, u, \bar{u}; \tilde{u}] = \alpha(\xi)[\bar{u}(\xi) + u(\xi)]\tilde{u}(\xi) + \tilde{u}(\xi)\int_0^1 \beta(\xi, \xi_1)\bar{u}(\xi_1)d\xi_1 + u(\xi)\int_0^1 \beta(\xi, \xi_1)\tilde{u}(\xi_1)d\xi_1 + \int_0^1 \gamma(\xi, \xi_1)[\bar{u}(\xi_1) + u(\xi_1)]\tilde{u}(\xi_1)d\xi_1 + \int_0^1\int_0^1 \epsilon(\xi, \xi_1, \xi_2)[\tilde{u}(\xi_1)\bar{u}(\xi_2) + u(\xi_1)\tilde{u}(\xi_2)]d\xi_1 d\xi_2.$$

The differential is obtained by putting $\bar{u} = u$ and is

$$\mathfrak{A}[\xi, u, u; \tilde{u}] = 2\alpha(\xi)u(\xi)\tilde{u}(\xi) + \tilde{u}(\xi)\int_0^1 \beta(\xi, \xi_1)u(\xi_1)d\xi_1 + u(\xi)\int_0^1 \beta(\xi, \xi_1)\tilde{u}(\xi_1)d\xi_1 + 2\int_0^1 \gamma(\xi, \xi_1)u(\xi_1)\tilde{u}(\xi_1)d\xi_1 + 2\int_0^1\int_0^1 \epsilon(\xi, \xi_1, \xi_2)\tilde{u}(\xi_1)u(\xi_2)d\xi_1 d\xi_2.$$

Consider now the first differential $\mathfrak{A}[\xi, u, \tilde{u}]$ of the integral power form $\mathfrak{A}[\xi, u]$ of the mth order. If one puts in this $u + \hat{u}$ for u and arranges the result according to the powers of $\hat{u}$, then the sum of the terms of the first power in $\hat{u}$ is called the second differential of $\mathfrak{A}[\xi, u]$ and is denoted by $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$. In a similar way one could define the higher differentials of $\mathfrak{A}[\xi, u]$. One may easily deduce the formula

$$(14) \qquad \mathfrak{A}[\xi, u + \tilde{u}] = \mathfrak{A}[\xi, u] + \frac{\mathfrak{A}[\xi, u; \tilde{u}]}{1!} + \frac{\mathfrak{A}[\xi, u; \tilde{u}, \tilde{u}]}{2!} + \cdots$$

with the analogues of Euler's formulas

$$\mathfrak{A}[\xi, u; u] = m\mathfrak{A}[\xi, u], \qquad \mathfrak{A}[\xi, u; u, u] = m(m-1)\mathfrak{A}[\xi, u], \cdots$$

which may be obtained from the obvious relation

$$\mathfrak{A}[\xi, cu] = c^m\mathfrak{A}[\xi, u],$$

where c is an arbitrary constant.

If in $\mathfrak{A}[\xi, u]$ all the coefficients are replaced by their absolute values and $u(\xi)$ by 1, one obtains a function of ξ whose maximum is called the height (Höhe) of $\mathfrak{A}[\xi, u]$. In a similar way one may define the height of $\mathfrak{A}[\xi, u; \tilde{u}]$, $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$, etc., by replacing the coefficients by their absolute values and $u, \tilde{u}, \hat{u}, \cdots$ by 1 and then taking the maximum with respect to ξ. If μ is the height of $\mathfrak{A}[\xi, \mu]$, then by Euler's relations $m\mu$ is the height of $\mathfrak{A}[\xi, u; \tilde{u}]$, $m(m-1)\mu$ is the height of $\mathfrak{A}[\xi, u; \tilde{u}, \hat{u}]$ etc.

One may now with Schmidt consider infinite sums of integral power forms (integral power series), viz.,

$$\mathfrak{P}[\xi, u] = \mathfrak{A}_0[\xi, u] + \mathfrak{A}_1[\xi, u] + \mathfrak{A}_2[\xi, u] + \cdots,$$

where $\mathfrak{A}_p[\xi, u]$ is an integral power form of the pth order. Let μ_p represent the heights of $\mathfrak{A}_p[\xi, u]$ and consider the series

$$\mu_0 + \mu_1 x + \mu_2 x^2 + \cdots.$$

If the radius of convergence R of this power series is different from zero,

then the integral power series is said to be *regular* for $|u(\xi)| \leqq R$. If $|u(\xi)| \leqq r < R$, then the above power series is dominated by

$$P(r) = \mu_0 + \mu_1 r + \mu_2 r^2 + \cdots$$

since $|\mathfrak{A}_p[\xi, u]| \leqq \mu_p r^p$. If $|u(\xi)| \leqq r$ and $|\tilde{u}(\xi)| \leqq s$, $r + s < R$, one has immediately the inequalities

$$(15) \qquad \begin{gathered} |\mathfrak{A}_p[\xi, u]| \leqq \mu_p r^p, \qquad |\mathfrak{A}_p[\xi, u; \tilde{u}]| \leqq p\mu_p r^{p-1} s, \\ |\mathfrak{A}_p[\xi, u; \tilde{u}, \tilde{u}]| \leqq p(p-1)\mu_p r^{p-2} s^2. \end{gathered}$$

Moreover Kowalewski shows by means of these that the following expansion is valid:

$$(16) \quad \mathfrak{P}[\xi, u + \tilde{u}] = \mathfrak{P}[\xi, u] + \mathfrak{P}[\xi, u; \tilde{u}] + \tfrac{1}{2}\mathfrak{P}[\xi, u; \tilde{u}, \tilde{u}] + \cdots,$$

where

$$\begin{aligned} \mathfrak{P}[\xi, u; \tilde{u}] &= \mathfrak{A}_1[\xi, u; \tilde{u}] + \mathfrak{A}_2[\xi, u; \tilde{u}] + \cdots, \\ \mathfrak{P}[\xi, u; \tilde{u}, \tilde{u}] &= \mathfrak{A}_2[\xi, u; \tilde{u}, \tilde{u}] + \mathfrak{A}_3[\xi, u; \tilde{u}, \tilde{u}] + \cdots, \text{ etc.,} \end{aligned}$$

From the inequalities (15) one obtains directly the dominance relation

$$(17) \quad \sum_{p,q}^{0\infty} \frac{1}{p!} |\mathfrak{A}_{p+q}[\xi, u; \overset{1}{\tilde{u}}\, \overset{2}{\tilde{u}} \cdots \overset{p}{\tilde{u}}]| \ll \sum_{p,q}^{0\infty} \frac{(p+q)\cdots(q+1)}{p!} \mu_{p+q} r^q s^p,$$

which may also be written with the help of (16) supposing $r + s < R$,

$$(18) \qquad |\mathfrak{P}[\xi, u + \tilde{u}]| \ll P(r + s).$$

Furthermore

$$(18') \qquad \begin{aligned} |\mathfrak{P}[\xi, u; \tilde{u}]| &\ll P'(r)s, \\ |\mathfrak{P}[\xi, u; \tilde{u}, \tilde{u}]| &\ll P''(r)s^2, \qquad \text{etc.,} \end{aligned}$$

where $P'(r)$, $P''(r)$, etc., are the first, second, etc., derived series of $P(r)$.

It is desired now to investigate the differential equation containing integral power series

$$(19) \qquad \begin{cases} \dfrac{\partial u(\xi, \tau)}{\partial \tau} = \mathfrak{P}[\xi, u], \\ u(\xi, 0) = 0 \qquad (\xi) \end{cases}$$

and see if the theorems proved in sections 1–4 are applicable. Kowalewski proves that such equations have solutions by a method very much analogous to that employed in proving the existence theorem for differential equations involving analytic functions. He shows in fact that the solution of

$$\frac{\partial u(\xi, \tau)}{\partial \tau} = \mathfrak{P}[\xi, u]$$

may be developed as a power series in τ with regular integral power series in u for coefficients.

It will be shown that the theory of the system (19) may be considered as a special instance of the theory described in this paper.

The following theorem is true.

THEOREM I′. *If* $\mathfrak{P}[\xi, u]$ *is a regular integral power series for* $\| u \| \leqq R$, *then the system* (19) *has a unique solution* $v(\xi, \tau)$ *continuous in*

$$(B') \qquad 0 \leqq \xi \leqq 1, \quad |\tau| \leqq \rho, \qquad \left(\rho = \frac{\beta}{P(R)}, \quad \beta = \frac{R}{3}\right).$$

The set (A') is now given by

$$(A') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau| < \infty, \qquad \| u \| \leqq \beta.$$

In the first place, since each term of the integral power series is a continuous function of ξ, and since the series is uniformly convergent with regard to ξ for each fixed u of (A'), it is clear that hypothesis (H_1) of Theorem I is satisfied. To prove (H_2), replace in (16) the variables $u, \bar{u}$ by $u, u' - u$ respectively, with u, u' both satisfying (A'). Then, for each such u and u', one has

$$(20) \quad \mathfrak{P}[\xi, u'] - \mathfrak{P}[\xi, u] = \mathfrak{P}[\xi, u; u' - u] + \tfrac{1}{2}\mathfrak{P}[\xi, u; u' - u, u' - u] + \cdots$$

so that by the inequalities (18′), one finds that the right-hand side of the last equation is dominated by

$$s\left[P'(\beta) + \frac{s}{2}P''(\beta) + \cdots\right]$$

where $s = \| u' - u \|$. Hence, since $s = \| u' - u \| \leqq 2\beta$ and $\gamma + s \leqq 3\beta < \kappa$, it follows from (17) and (18) that

$$|\mathfrak{P}[\xi, u'] - \mathfrak{P}[\xi, u]| \leqq sP'(\beta + s) \leqq P'(R) \| u' - u \|.$$

Thus the hypotheses of Theorem I are satisfied. From the statements just preceding equation (15), it is clear that the value $\gamma = P(R)$ is an upper bound for $\mathfrak{P}[\xi, u]$.

The theorem on the continuity with regard to the initial elements reads precisely as Theorem III and will not be stated here.

THEOREM IV′. *If* $\mathfrak{P}[\xi, u]$ *is a regular integral power series for* $\| u \| \leqq R$, *then in the set*

$$(B_0') \qquad 0 \leqq \xi \leqq 1, \qquad |\tau - \tau_0| \leqq \rho, \qquad |\tau_0| \leqq \delta, \qquad \| u_0 \| \leqq \delta,$$

the solution $v[\xi, \tau, \tau_0, u_0]$ *has a difference function* $b[\xi, \tau, \tau_0, u_0, \bar{\tau}, \bar{\tau}_0, \bar{u}_0; \bar{\tau}, \bar{\tau}_0\bar{u}_0]$

The set corresponding to (A_0) is here given by

$$(A_0') \quad 0 \leqq \xi \leqq 1, \qquad -\infty < \tau < \infty, \qquad \| u \| \leqq \beta + \delta, \qquad (\beta + \delta < R/3).$$

It is requisite to show that the hypotheses of Theorem IV are satisfied. The proof that the hypotheses of Theorem I are satisfied in (A_0') is the same as that given in Theorem I'. To show that $\mathfrak{P}[\xi, u]$ has a difference function one makes use of the expansion (20). Consider the right-hand side with the last argument $u' - u$ of each term replaced by $\bar{u}$,

$$a[\xi, u, u'; \bar{u}] = \mathfrak{P}[\xi, u; \bar{u}] + \tfrac{1}{2}\mathfrak{P}[\xi, u; u' - u, \bar{u}] + \cdots.$$

The linearity of the functional a in $\bar{u}$ is clear from the definitions of $\mathfrak{P}[\xi, u; \bar{u}]$, $\mathfrak{P}[\xi, u; u' - u, \bar{u}]$, etc. Furthermore, the functional a is continuous for every continuous function $\bar{u}$. For, set $s = ||u' - u||$ and $S = ||\bar{u}||$; then, by inequalities analogous to (15), (17) and (18'), it follows that

$$|\mathfrak{P}[\xi, u; \bar{u}]| \ll P'(\beta)S \ll P'(R)S,$$
$$|\mathfrak{P}[\xi, u; u' - u, \bar{u}]| \ll P''(\beta)Ss \ll P''(R)Ss, \text{ etc.},$$

Hence, the series for $a[\xi, u, u'; \bar{u}]$ is dominated by $P'(\beta + s)S$ or $P'(R)S$; i.e., for a fixed $\bar{u}$ it converges uniformly with respect to its other arguments and consequently represents a continuous functional for each fixed $\bar{u}$. The constant μ associated with the difference function is $P'(R)$.

It remains to prove that a has the property (1'). In the first place it can readily be shown by a method of proof similar to that used in the proof of Lemma 1, section 2, that, if each of a sequence of functionals has the property (1'), and the sequence converges uniformly in the set under consideration, then the limit functional also has the property (1'). Since the above series for a does indeed converge uniformly, it suffices to prove that the difference function of (13) has the property (1'). But it follows from the definition of a difference function and from the form of (13) that the difference function of (13) is a sum of integrals of the same type except that in the integrand product there occur two functions u, $\bar{u}$ instead of a single function u. That integrals of this last type have property (1'), follows very readily from the fact that the integrals are continuous functionals of u and $\bar{u}$.

THE FACTORIZATION OF THE RATIONAL PRIMES IN A CUBIC DOMAIN.

By G. E. Wahlin.

I. Introduction.

The object of this paper is to determine a set of functions of the coefficients of a cubic equation whose character, with respect to the prime modulus p, completely determines the factorization of p in the cubic domain defined by a root of the equation.

Since every cubic equation can, by a linear transformation, be transformed into the form $x^3 + Cx + D = 0$, where C and D are rational integers, there is no loss in assuming the cubic equation in the following discussion to be of this form. Moreover if $C = p^\lambda \cdot C_1$ and $D = p^\mu \cdot D_1$ where C_1 and D_1 are prime to p, when $\lambda \geqq 2$ and $\mu \geqq 3$ the roots of the equation may be divided by a power of p and the equation thus further reduced. We shall therefore throughout the following pages suppose that $0 \leqq \lambda < 2$ or $0 \leqq \mu < 3$.

We shall denote $x^3 + Cx + D$ by $F_3(x)$ and $3Cx^2 + 9Dx - C^2$ by $F_2(x)$. The discriminant of $F_3(x) = 0$ is $-27D^2 - 4C^3$ and shall be denoted by Δ_3. The discriminant of $F_2(x) = 0$ is $-3\Delta_3$ and shall be denoted by Δ_2.

Let μ_1 and μ_2 be the roots of

$$F_2(x) = 0. \tag{1}$$

Since $F_3(x)$ is supposed to be irreducible $\Delta_3 \neq 0$ and hence $\Delta_2 \neq 0$ and $\mu_1 \neq \mu_2$.

If $\mu_1 = (-9D - \sqrt{\Delta_2})/6C$ and $\mu_2 = (-9D + \sqrt{\Delta_2})/6C$, we shall write $(6C\mu_2)^n = \psi_n(-9D, \Delta_2) + \varphi_n(-9D, \Delta_2)\sqrt{\Delta_2}$ where ψ_n and φ_n are polynomials. Hence $(6C\mu_2)^n - (6C\mu_1)^n = 2\varphi_n(-9D, \Delta_2)\sqrt{\Delta_2}$.

We shall next apply, to the cubic $F_3(x) = 0$, the non-singular transformation

$$x = \frac{\mu_1 y + \mu_2}{y + 1}.$$

After simplifying the new equation has the form $F_3(\mu_1)y^3 + F_3(\mu_2) = 0$. Since

$$F_3(x) = F_2(x)\left(x - \frac{3D}{C}\right) \cdot \frac{1}{3C} - \frac{\Delta_3}{3C^2}x \tag{2}$$

and μ_1 and μ_2 are roots of (1), we have $F_3(\mu_i) = -(\Delta_3/3C^2)\mu_i$ $(i = 1, 2)$

and hence the transformed cubic may be written in the form

$$y^3 + \frac{\mu_2}{\mu_1} = 0 \tag{3}$$

and the roots of this are $\rho^i \sqrt[3]{-(\mu_2/\mu_1)}$ $(i = 0, 1, 2)$, where ρ is a primitive cube root of unity. The roots of $F_3(x) = 0$ are therefore

$$\alpha_i = \frac{\mu_1 \rho^i \sqrt[3]{-\frac{\mu_2}{\mu_1}} + \mu_2}{\rho^i \sqrt[3]{-\frac{\mu_2}{\mu_1}} + 1} \quad (i = 0, 1, 2),$$

and in simplified form $\alpha_0 = -\mu_1^{1/3} \cdot \mu_2^{1/3}(\mu_1^{1/3} + \mu_2^{1/3})$.

The above implies that $C \neq 0$, but this does not invalidate the applications which shall be made of it, because it will be seen that in all cases where it is used C is necessarily different from zero.

The theory underlying the following development is that of the application of the p-adic numbers to the study of the algebraic numbers.*

II. EISENSTEINIAN FUNCTIONS.

A polynomial of the form

$$E(x) = x^n + p^{e_1}a_1x^{n-1} + p^{e_2}a_2x^{n-2} + \cdots + p^{e_{n-1}}a_{n-1}x + p^{e_n}a_n,$$

in which all the coefficients except that of the highest power of x are divisible by the prime p is called an Eisensteinian function. Eisenstein's theorem, that every such polynomial in which $e_n = 1$ is irreducible, is well known. O. Perron† has generalized this theorem as follows. The algebraic equation

$$x^n + p^{[e/n]+1}a_1x^{n-1} + p^{[2e/n]+1}a_2x^{n-2} + \cdots + p^{[(n-1)e/n]+1}a_{n-1}x + p^e a_n = 0,$$

where $a_1, a_2, \cdots a_n$ are arbitrary integers and a_n prime to p and e prime to n is irreducible. The proof of this theorem consists in showing that, in the domain defined by a root of the given equation, p is the nth power of a prime ideal and hence the domain must be of degree n. We note here that this method is also sufficient to show the irreducibility in $k(p)$.

In the "Theorie der Algebraischen Zahlen" Hensel shows that every factor of an Eisensteinian function in $k(p)$ is an Eisensteinian function and hence the number of factors cannot exceed the exponent of p in the last term.

From Perron's theorem we can conclude the following fact:

* Hensel, "Theorie der Algebraischen Zahlen." Author, *Transactions Am. Math. Soc.*, Vol. 16. A new development of the theory for quadratic domains was published by Hensel in *Crelle's Journal*, Vol. 144. In the author's paper, here referred to, he gives an extension of this to the general case. Of importance, in the following pages, is the isomorphism between the two domains $k(\mathfrak{p}_i, \alpha)$ and $k(p, \alpha_i^{(r)})$ discussed in the author's paper.

† *Mathematische Annalen*, Vol. 60, Theorem I.

A. If in $E(x)$ $e_i \geqq e_n$, a_n is prime to p and e_n is prime to n, then, in the domain defined by a root of $E(x) = 0$, p is the nth power of a prime divisor and $E(x)$ is irreducible in $k(p)$.

We shall prove the following fact:

B. If in $E(x)$ $e_i \geqq e_{n-1}$, $e_{n-1} \leqq e_n/2$, $n > 2$ and a_{n-1} and a_n are prime to p, then $E(x)$ has a linear factor in $k(p)$.

Since a_{n-1} and a_n are prime to p there exists a c such that $a_{n-1}c + a_n \equiv 0 \mod p$. Then

$$E(p^{e_n-e_{n-1}}c) = p^{(e_n-e_{n-1})n}c^n + p^{(e_n-e_{n-1})(n-1)+e_1}a_1c^{n-1} + \cdots + p^{(e_n-e_{n-1})2+e_{n-2}}a_{n-2}c^2 + p^{e_n}(a_{n-1}c + a_n).$$

Since $e_{n-1} \leqq e_n/2$, $e_n - e_{n-1} \geqq e_n/2$; and since $n > 2$, $(e_n - e_{n-1})n > e_n$. Since $e_i \geqq e_{n-1}$ for $i \leqq n-2$, $(e_n - e_{n-1})(n-i) + e_i > e_n - e_{n-1} + e_i > e_n$; and since $a_{n-1}c + a_n \equiv 0 \mod p$, we see that

$$E(p^{e_n-e_{n-1}}c) \equiv 0 \mod p^{e_n+1}.$$

If we next form $E'(p^{e_n-e_{n-1}}c)$, the first term is divisible by $p^{(e_n-e_{n-1})(n-1)}$ and since $e_n - e_{n-1} \geqq n/2$ and $n - 1 \geqq 2$, it is divisible by p^{e_n}. The following terms up to and including the next to the last are divisible by powers of p whose exponents are, in the successive terms, greater than $e_1, e_2, \cdots e_{n-2}$ and hence greater than e_{n-1}. The last term is divisible only by $p^{e_{n-1}}$. Hence $E(x) = 0$ (p) has a solution in $k(p)$,* and in this domain $E(x) = (x - p^{e_n-e_{n-1}}\cdot\gamma)\cdot Q(x)$.

C. If in B, $e_{n-1} = 1$, $Q(x)$ is seen to be irreducible by Eisenstein's theorem because it is an Eisensteinian function whose last term contains only the first power of p. If in this case $E(x) = 0$ is irreducible in $k(1)$, p is, in the domain defined by a root of this equation, the product of a prime divisor by the $n-1$th power of another prime, as is seen by A and the isomorphism referred to in the note in the introduction.

We are now ready to consider the factorization of p in the cubic domain $k(\alpha_i)$. I shall denote the prime divisors by $\mathfrak{p}$. As there is no danger of any ambiguity regarding the degrees of the various prime divisors, I shall make no indications of them in the statement of results.

We shall first consider the case when $p > 3$ and is not a divisor of Δ_3.

III. Δ_2 A QUADRATIC RESIDUE MOD p.

Since $\Delta_2 = 81D^2 + 12C^3$ it may happen in this case that $C \equiv 0 \mod p$ and we shall consider this possibility first.

Since p is not a factor of Δ_3 it is not a factor of the discriminant of $k(\alpha_i)$ and is therefore not divisible by a power of a prime divisor. Moreover the

* Hensel, A.Z., p. 71, bottom.

number of factors of $F_3(x)$ in $k(p)$ is the same as the number of factors of $F_3(x)$ mod p.* When $C \equiv 0 \bmod p$, $F_3(x) \equiv x^3 + D \bmod p$ and we see that when $p \equiv 1 \bmod 3$, if $-D$ is a cubic residue mod p, $F_3(x) \equiv 0 \bmod p$ has three solutions in $k(1)$, and when $-D$ is not a cubic residue the same congruence has no rational solution. If $p \equiv -1$, $-D$ is always a cubic residue mod p but in this case $F_3(x) \equiv 0 \bmod p$ has only one solution in $k(1)$. Hence when $p \equiv +1 \bmod 3$, if $-D$ is a cubic residue mod p, $F_3(x)$ is in $k(p)$ the product of three linear factors, if $-D$ is not a cubic residue mod p, $F_3(x)$ is irreducible in $k(p)$ and when $p \equiv -1 \bmod 3$, $F_3(x)$ is the product of a linear and a quadratic factor in $k(p)$.

Since the cubic character of $-D$ is the same as the cubic character of Δ_3 we can conclude that, when $p \equiv +1 \bmod 3$ if $\Delta_3^{(p-1)/3} \equiv 1 \bmod p$, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$, and if $\Delta_3^{(p-1)/3} \not\equiv 1 \bmod p$, then $p \sim \mathfrak{p}$; and when $p \equiv -1 \bmod 3$, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

If $C \not\equiv 0 \bmod p$ the roots of (1) are units with respect to p. Since Δ_2 is a quadratic residue $F_2(x)$ is reducible in $k(p)$. Hence, if Δ_2 is not a square, $p \sim \mathfrak{p}' \cdot \mathfrak{p}''$ in $k(\mu_1)$ and if Δ_2 is a square, $k(\mu_1) = k(1)$ and $p \sim \mathfrak{p}'$.

Let us now consider the equation

$$y^3 + \frac{\mu_2}{\mu_1} = 0 \qquad (\mathfrak{p}'). \tag{4}$$

If $p \equiv -1 \bmod 3$ this will always have a solution in $k(\mathfrak{p}', \mu_1)$ but if $p \equiv 1 \bmod 3$ it has a solution when and only when $(\mu_2/\mu_1)^{(n-1)/3} \equiv 1 \bmod \mathfrak{p}'$. But this condition is equivalent to $\mu_2^{(p-1)/3} - \mu_1^{(p-1)/3} \equiv 0 \bmod \mathfrak{p}'$ and hence according to the notation in the introduction it is seen to be equivalent to $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod \mathfrak{p}'$. But $\varphi_{(p-1)/3}(-9D, \Delta_2)$ is a rational integer and is therefore divisible by $\mathfrak{p}'$ when and only when it is divisible by p. Hence in order that (4) shall have a solution in $k(\mathfrak{p}', \mu_1)$ it is necessary and sufficient, either that $p \equiv -1 \bmod 3$, or that $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$.

Every number of $k(\mathfrak{p}', \mu_1)$ is for the domain of $\mathfrak{p}'$ equal to a number of $k(p)$ and hence there exist rational p-adic numbers m_1 and m_2 such that $\mu_1 = m_1$ $(\mathfrak{p}')$ and $\mu_2 = m_2$ $(\mathfrak{p}')$ and if (4) has a solution this is also for the domain of $\mathfrak{p}'$ equal to a rational p-adic number b.

Let $\overline{\mathfrak{p}}'$ be that prime divisor of $\mathfrak{p}'$ in $k[\mu_1, \sqrt[3]{-(\mu_2/\mu_1)}]$ corresponding to the linear factor of (4). Then $\sqrt[3]{-(\mu_2/\mu_1)} = b$ $(\overline{\mathfrak{p}}')$ and $\overline{\mathfrak{p}}'$ being a factor of $\mathfrak{p}'$, $\mu_1 = m_1$ and $\mu_2 = m_2$ $(\overline{\mathfrak{p}}')$. Hence, if we put $a_0 = (m_1 b + m_2)/(b + 1)$ we see that $\alpha_0 = a_0$ $(\overline{\mathfrak{p}}')$ and hence $F_3(a_0) = 0$ $(\overline{\mathfrak{p}}')$. But from this it follows that $F_3(x) = (x - a_0)Q(x)$ $(\overline{\mathfrak{p}}')$, and since the coefficients of both members of this equation are rational p-adic numbers it is evident that the two members are equal for the domain of $\overline{\mathfrak{p}}'$ when and only when they are equal

* Hensel, A.Z., p. 68.

for the domain of p, and hence

$$F_3(x) = (x - a_0) \cdot Q(x) \quad (p) \tag{5}$$

if (4) has a solution in $k(\mathfrak{p}', \mu_1)$.

If however (4) does not have a solution in $k(\mathfrak{p}', \mu_1)$, it is irreducible in this domain and the same is then also true of $F_3(x)$ and hence a fortiori it is irreducible in $k(p)$.

In regard to $Q(x)$ we observe that since its discriminant differs from Δ_3 only by a square factor it is reducible when Δ_3 is a quadratic residue mod p and is irreducible when Δ_3 is not a quadratic residue mod p. But $\Delta_2 = -3\Delta_3$ is a quadratic residue mod p and hence Δ_3 is a quadratic residue when and only when -3 is a quadratic residue and hence when and only when $p \equiv 1 \bmod 3$.

Hence when $p \equiv 1 \bmod 3$ if $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$, then $F_3(x)$ is in $k(p)$ the product of three linear factors and if $\varphi_{(p-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$, $F_3(x)$ is irreducible in $k(p)$; and when $p \equiv -1 \bmod 3$, $F_3(x)$ is in $k(p)$ the product of a linear and a quadratic factor.

Regarding the factors of p in $k(\alpha_i)$ we can conclude that when $p \equiv 1 \bmod 3$ and $\varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$; when $p \equiv 1 \bmod 3$ and $\varphi_{(p-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$, then $p \sim \mathfrak{p}$; and when $p \equiv -1 \bmod 3$, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

IV. Δ_2 NOT A QUADRATIC RESIDUE MOD p.

When Δ_2 is not a quadratic residue mod p, $F_2(x)$ is irreducible in $k(p)$ and hence in $k(\mu_1)$ p is a prime of the second degree. Using the same method as in III we see that the equation

$$y^3 + \mu_2/\mu_1 = 0 \qquad (p) \tag{6}$$

has a solution in $k(p, \mu_1)$ when and only when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$ because in this case it is necessary and sufficient that $(\mu_2/\mu_1)^{(p^2-1)/3} \equiv 1 \bmod p$.

Let us suppose that a solution exists and that $\bar{\mathfrak{p}}$ is the prime divisor of p in $k[\mu_1\sqrt[3]{-(\mu_2/\mu_1)}]$ corresponding to the linear factor of (6). Then there exists an a_0 in $k(p, \mu_1)$ such that $\alpha_0 = a_0(\bar{\mathfrak{p}})$ and hence as above $F_3(x) = (x - a_0)Q(x)$ $(\bar{\mathfrak{p}})$. Again since the coefficients of both members of this equation belong to $k(p, \mu_1)$ in which p is a prime, we see that it is true for the domain of $\bar{\mathfrak{p}}$ when and only when it is true for the domain of p. Hence

$$F_3(x) = (x - a_0)\overline{Q}(x) \quad (p). \tag{7}$$

If a_0 is a rational number, $F_3(x)$ has a linear factor in $k(p)$. If a_0 is a quadratic number and a_0' its conjugate since $F_3(a_0) = 0$ (p), we know that $F_3(a_0') = 0$ (p). Hence $F_3(x) = (x - a_0) \cdot (x - a_0')(x - a)$ (p) and since

the coefficients of $(x - a_0)(x - a_0')$ are rational a must be rational and again we see that $F_3(x)$ has a linear factor in $k(p)$.

If (6) has no solution in $k(p, \mu_1)$ in the same way as in III, we conclude that $F_3(x)$ is irreducible in $k(p)$.

Considering the case when $F_3(x)$ is reducible let us write $F_3(x) = (x - a) \cdot Q(x)$. As in III we conclude that $Q(x)$ is reducible when and only when Δ_3 is a quadratic residue mod p and since now Δ_2 is not a quadratic residue this is the case when and only when -3 is not a quadratic residue and hence $p \equiv -1 \bmod 3$.

Hence when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$ and $p \equiv -1 \bmod 3$, $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$; when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$ and $p \equiv 1 \bmod 3$, $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$; and when $\varphi_{(p^2-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$ $p \sim \mathfrak{p}$.

V. $\Delta_2 \equiv 0 \bmod p$, $p > 3$.

We shall next consider the factorization of those primes which are greater than 3 and are factors of the discriminant of $F_3(x)$. We shall write $\Delta_3 = p^s \Delta_3'$ where we suppose that Δ_3' is prime to p. As stated in I we shall put $C = p^\lambda \cdot C_1$, and $D = p^\mu D_1$. Since $s > 0$ if λ or μ is zero the other must be zero also and we shall consider this possibility first. From (2) we have

$$F_3(x) \equiv F_2(x) \cdot \left(x - \frac{3D}{C}\right) \cdot \frac{1}{3C} \bmod p^s \tag{8}$$

and the resultant of the two factors $R = R[F_2(x), x - (3D/C)] = 1/C \cdot (54D^2 - C^3)$. Since $\Delta_3 = 27D^2 + 4C^3 \equiv 0 \bmod p$, $27D^2 \equiv -4C^3 \bmod p$ and hence $R \equiv -9C^2 \bmod p$ and is therefore prime to p. Hence $F_3(x)$ is reducible in $k(p)$.*

We shall again write $F_3(x) = (x - a)Q(x)$ (p) and studying the quadratic factor in the same way as in III we can conclude that when s is even and Δ_3' is a quadratic residue, $Q(x)$ is reducible in $k(p)$, and when s is even and Δ_3' is not a quadratic residue mod p, it is irreducible.

Using the isomorphism mentioned in the note in I we can conclude that when s is even if Δ_3' is a quadratic residue mod p, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$ in $k(\alpha_i)$ and when Δ_3' is not a quadratic residue mod p, then $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$ in $k(\alpha_i)$. When s is odd, $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$ in $k(\alpha_i)$.

If however $\lambda > 0$ it follows that $\mu > 0$ and we need only consider the cases when $\lambda < 2$ or $\mu < 3$. We shall take up the two cases (a) $\lambda \geqq \mu$ and (b) $\lambda < \mu$.

(a) Since $\lambda \geqq \mu > 0$ by the restrictions imposed on λ and μ we conclude that in this case $\mu = 1$ or 2 and hence by II, A we know that $F_3(x)$ is irreducible in $k(p)$ and $p \sim \mathfrak{p}^3$ in $k(\alpha_i)$.

* Hensel, A.Z., p. 71.

(*b*) Since now $\lambda < \mu$ we see that this is possible only when $\lambda = 1$ and hence, by II, *B* and *C*, we can conclude that $F_3(x)$ is in $k(p)$ the product of a linear and a quadratic factor and $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$ in $k(\alpha_i)$.

VI. $p = 3$.

We shall next consider the factorization of 3 in $k(\alpha_i)$. As before we shall write $C = 3^\lambda C_1$, $D = 3^\mu D_1$ and $\Delta_3 = 3^s \Delta_3'$. If $\lambda = 0$, $s = 0$ and we need only consider $F_3(x)$ with respect to the modulus 3. There are six possible forms as follows:

$$\begin{aligned}
&F_3(x) \equiv x^3 + x \equiv x(x^2+1); && \Delta_3 \equiv -1 \bmod 3,\\
&F_3(x) \equiv x^3 + 2x \equiv x(x-1)(x+1); && \Delta_3 \equiv 1 \bmod 3,\\
&F_3(x) \equiv x^3 + x + 1 \equiv (x-1)(x^2+x+2); && \Delta_3 \equiv -1 \bmod 3,\\
&F_3(x) \equiv x^3 + 2x + 1 \text{ irreducible}; && \Delta_3 \equiv 1 \bmod 3,\\
&F_3(x) \equiv x^3 + x + 2 \equiv (x-2)(x^2+2x+2); && \Delta_3 \equiv -1 \bmod 3,\\
&F_3(x) \equiv x^3 + 2x + 2 \text{ irreducible}; && \Delta_3 \equiv 1 \bmod 3.
\end{aligned}$$

From a consideration of this table we note that when $\Delta_3 \equiv 1 \bmod 3$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$ if $D \equiv 0 \bmod 3$ and $3 \sim \mathfrak{p}$ if $D \not\equiv 0 \bmod 3$; and when $\Delta_3 \equiv -1 \bmod 3$, then $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

If $s > 0$ it is necessary that $\lambda > 0$. As before we need only consider the cases when $\lambda < 2$ or $\mu < 3$. These we shall consider as follows (*a*) $\lambda = 1$, $\mu = 0$; (*b*) $\lambda > 1$, $\mu = 0$; (*c*) $\lambda \geqq \mu$; (*d*) $\lambda < \mu$.

(*a*) When $\lambda = 1$, $\mu = 0$, $\Delta_3 = -27(D^2 + 4C_1^3)$ and $s \geqq 3$. The function $F_2(x)$ is $9C_1x^2 + 9Dx - 9C_1^2$ and μ_1 and μ_2 are roots of

$$C_1x^2 + Dx - C_1^2 = 0. \tag{9}$$

In place of the equation (2) we have now

$$F_3(x) = (C_1x^2 + Dx - C_1^2)\left(x - \frac{D}{C_1}\right) \cdot \frac{1}{C_1} - \frac{\Delta_3}{27C_1}x \tag{10}$$

and hence when $s > 3$,

$$F_3(x) \equiv (C_1x^2 + Dx - C_1^2)\left(x - \frac{D}{C_1}\right) \cdot \frac{1}{C_1} \bmod 3^{s-3}. \tag{11}$$

Since in this case (i.e., $s > 3$) $D^2 + 4C_1^3 \equiv 0 \bmod 3^{s-3}$,

$$D^2 \equiv -4C_1^3 \bmod 3^{s-3}. \tag{12}$$

Therefore

$$F_3'\left(\frac{D}{C_1}\right) \equiv \frac{3D^2 + 3C_1^3}{C_1^2} = \frac{3}{C_1^2}(D^2 + C_1^3) \equiv -9C_1 \bmod 3^{s-3}$$

and if $s > 6$, $F_3'(D/C_1)$ is divisible only by 3^2. $F_3''(D/C_1)/2! = 3(D/C_1)$ and is divisible only by 3. $F_3'''(D/C_1)/3! = 1$ and is prime to 3. Hence using the theorem on pages 73 and 74 of Hensel's Theorie der Algebraischen

Zahlen we see that when $s > 6$, $F_3(x)$ has a linear factor in $k(3)$, and shall write

$$F_3(x) = (x - a)Q(x) \cdot (3).$$

Regarding the quadratic factor we can conclude in the same way as in V that it is reducible when and only when s is even and $\Delta_3' \equiv 1 \bmod 3$. By the isomorphism previously used we then see that if s is even, when $\Delta_3' \equiv 1 \bmod 3$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$, and when $\Delta_3' \equiv -1 \bmod 3$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$; and that when s is odd, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

If $s = 6$, let us consider first the case when Δ_3' is not a quadratic residue mod 3. Then

$$\Delta_3' = -\frac{D^2 + 4C_1^3}{27} \equiv -1 \bmod 3 \tag{13}$$

and since $D^2 + 4C_1^3 \equiv 0 \bmod 3$ and $D^2 \equiv 4 \equiv 1 \bmod 3$, $C^3 \equiv -1 \bmod 3$ and we can write (13) in the form

$$-\frac{D^2 + 4C_1^3}{27} \equiv C_1^3 \bmod 3$$

whence $-D^2 - 4C_1^3 \equiv 27C_1^3 \bmod 81$ or

$$D^2 \equiv -31C_1^3 \bmod 81. \tag{14}$$

Now

$$F_3\left(-\frac{2D}{C_1}\right) = -\frac{D}{C_1^3}(8D^2 + 5C_1^3) \equiv 243D \equiv 0 \bmod 81$$

by (14).

$$F_3'\left(-\frac{2D}{C_1}\right) = \frac{3}{C_1^2}(4D^2 + C_1^3) \equiv -123 \cdot 3C_1 \bmod 81,$$

and is therefore divisible only by 3^2.

$$F_3''\left(-\frac{2D}{C_1}\right)\Big/2! = -\frac{6D}{C_1}$$

and is divisible only by 3.

$$F_3'''\left(-\frac{2D}{C_1}\right)\Big/3! = 1$$

and is prime to 3. In the same way as above we can conclude that $F_3(x)$ has a linear factor in $k(3)$ and since Δ_3' is not a quadratic residue the quadratic factor is irreducible in $k(3)$ and as above we conclude that $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$ in $k(\alpha_i)$.

If however $s = 6$ and Δ_3' is a quadratic residue mod 3, (10) shows that $F_3(D/C_1)$ is divisible by 3^3 but not by 3^4. Since now

$$\Delta_3' = -\frac{D^2 + 4C_1^3}{27} \equiv 1 \equiv -C_1^3 \bmod 3,$$

$-D^2 - 4C_1^3 \equiv -27C_1^3 \bmod 81$ and

$$D^2 \equiv 23C_1^3 \bmod 81. \tag{15}$$

Let us now form

$$F_3\left(3y + \frac{D}{C_1}\right) = 27y^3 + 27\frac{D}{C_1}y^2 + 9\left(\frac{D^2}{C_1^2} + C_1\right)y + F_3\left(\frac{D}{C_1}\right).$$

Dividing by 27 and equating to zero we have

$$y^3 + \frac{D}{C_1}y^2 + \frac{D^2 + C_1^3}{3C_1^2}y + F_3\left(\frac{D}{C_1}\right)\Big/27 = 0 \tag{16}$$

and evidently one root of this equation belongs to $k(\alpha_i)$. In this equation D/C_1 and $F_3(D/C_1)/27$ are known to be prime to 3. By (15) $D^2 + C_1^3 \equiv 24C_1^3 \bmod 81$ and hence $(D^2 + C_1^3)3C_1^2 \equiv 8C_1 \bmod 27$ and is therefore also prime to 3.

We shall next see that the left-hand member of (16) is irreducible mod 3. Suppose that it is not. It must then have a linear factor and hence there must exist an a such that

$$a^3 + \frac{D}{C_1}a^2 + \frac{D^2 + C_1^3}{3C_1^2}a + F_3\left(\frac{D}{C_1}\right)\Big/27 \equiv 0 \bmod 3.$$

Since the last term is prime to 3, a cannot be divisible by 3 and hence $a^3 \equiv a \bmod 3$ and $a^2 \equiv 1 \bmod 3$ and a must satisfy

$$\left(1 + \frac{D^2 + C_1^3}{3C_1^2}\right)a + \frac{D}{C_1} + F_3\left(\frac{D}{C_1}\right)\Big/27 \equiv 0 \bmod 3. \tag{17}$$

We have seen that $(D^2 + C_1^3)/3C_1^2 \equiv 8C_1 \bmod 27$ and since $C_1^3 \equiv -1 \bmod 3$, $C_1 \equiv -1 \bmod 3$ and hence $(D^2 + C_1^3)/3C_1^2 \equiv -8 \equiv 1 \bmod 3$.

$$\frac{D}{C_1} + F_3\left(\frac{D}{C_1}\right)\Big/27 = \frac{D}{27C_1^3}[27C_1^2 + D^2 + 4C_1^3].$$

By 15 we have $27C_1^2 + D^2 + 4C_1^3 \equiv 27C_1^2 + 27C_1^3 \bmod 81$ and hence

$$\frac{D}{27C_1^3}[27C_1^2 + D^2 + 4C_1^3] \equiv \frac{D}{C_1}(1 + C_1) \equiv 0 \bmod 3.$$

The congruence (17) therefore reduces to $2a \equiv 0 \bmod 3$ which is impossible since a is prime to 3. The left-hand member of (16) is therefore irreducible mod 3 and hence also in $k(3)$. Since this is so we know that 3 cannot be a divisor of the discriminant of (16) and hence also not a divisor of the discriminant of $k(\alpha_i)$ and is therefore not divisible by a power of a prime divisor and hence $3 \sim \mathfrak{p}$ in $k(\alpha_i)$.

If $s = 5$, $F_3(D/C_1)$ is divisible by 9 but not by 27. Multiplying (16) by 3 and forming the equation whose roots are the reciprocals of the roots of (16) we have

$$\frac{F_3(D/C_1)}{9} z^3 + \frac{D^2 + C_1^3}{C_1^2} z^2 + \frac{3D}{C_1} z + 3 = 0$$

and by II, A we see that this is irreducible in $k(p)$ and that in $k(\alpha_i)$ $3 \sim \mathfrak{p}^3$.

If $s = 4$, $F_3(D/C_1)$ is divisible only by 3 and hence the equation

$$F\left[y + \left(\frac{D}{C_1}\right)\right] = 0$$

is seen by II, A to be irreducible in $k(3)$ and again $3 \sim \mathfrak{p}^3$ in $k(\alpha_i)$.

If $s = 3$, $\Delta_3' = -D^2 - 4C_1^3$ and the discriminant of (9) is $-\Delta_3'$ Since $D^2 \equiv 1 \bmod 3$ and $\Delta_3' \not\equiv 0 \bmod 3$ it is necessary that $C \equiv 1 \bmod 3$ and hence $-\Delta_3' \equiv 2 \bmod 3$. The equation (9) is therefore irreducible in $k(3)$ and in $k(\mu_1)$ 3 is a prime of the second degree. As in IV we can show that $F_3(x)$ is reducible in $k(3)$ when and only when the equation $y^3 + (\mu_2/\mu_1) = 0$ (3) has a solution in $k(3, \mu_1)$. This equation has a solution when and only when the congruence $y^3 + (\mu_2/\mu_1) \equiv 0 \bmod 9$ has a solution in the given domain.*

If the given congruence has a solution b, we see that $b^9 \equiv -(\mu_2/\mu_1)^3 \bmod 27$ and since every integer of $k(3, \mu_1)$ satisfies the congruence $x^9 \equiv x \bmod 3$, we conclude that $b \equiv -(\mu_2/\mu_1)^3 \bmod 3$. Hence if $y^3 + (\mu_2/\mu_1) \equiv 0 \bmod 9$ has a solution, there must exist a c such that $[3c - (\mu_2/\mu_1)^3]^3 + \mu_2/\mu_1 \equiv 0 \bmod 9$. By expanding we find that the necessary and sufficient condition for the reducibility of $F_3(x)$ is $-(\mu_2/\mu_1)^9 + \mu_2/\mu_1 \equiv 0 \bmod 9$, or $(\mu_2/\mu_1)^8 \equiv 1 \bmod 9$ which reduces to $\varphi_8(-D, -\Delta_3') \equiv 0 \bmod 9$.

But $\varphi_8(-D, -\Delta_3') = -8D^7 + 56D^5\Delta_3' - 56D^3\Delta_3'^2 + 8D\Delta_3'^3$ and hence since $8D$ is relatively prime to 3, the necessary and sufficient condition is $\Delta_3'^3 - 7\Delta_3'^2D^2 + 7\Delta_3'D^4 - D^6 \equiv 0 \bmod 9$ or in another form

$$\left(\frac{\Delta_3'}{D^2}\right)^3 - 7\left(\frac{\Delta_3'}{D^2}\right)^2 + 7\left(\frac{\Delta_3'}{D^2}\right) - 1 \equiv 0 \bmod 9. \tag{18}$$

But 1 is the only rational solution of $x^3 - 7x^2 + 7x - 1 \equiv 0 \bmod 9$ and hence (18) is possible when and only when $\Delta_3'/D^2 \equiv 1 \bmod 9$, or $-D^2 - 4C_1^3 \equiv D^2 \bmod 9$. This reduces to $D^2 + 2C_1^3 \equiv 0 \bmod 9$ and since $C_1 \equiv 1 \bmod 3$, $C_1^3 \equiv 1 \bmod 9$ and hence this condition is equivalent to $D^2 + 2 \equiv 0 \bmod 9$.

Hence $F_3(x)$ is reducible in $k(3)$ when and only when $D^2 + 2 \equiv 0 \bmod 9$. Since $s = 3$, 3 is in $k(\alpha_i)$ divisible by a power of a prime divisor and hence $F_3(x)$ cannot have three linear factors in $k(3)$. We can therefore conclude that when $D^2 + 2 \equiv 0 \bmod 9$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$, and when $D^2 + 2 \not\equiv 0 \bmod 9$, $3 \sim \mathfrak{p}^3$.

* Author, *Crelle's Journal*, Vol. 145.

(*b*) Since in this case $\lambda > 1$, $\mu = 0$, we have $s = 3$ and again $F_3(x)$ is either irreducible or the product of a linear and a quadratic factor in $k(3)$. If $D^2 \equiv 1 \bmod 9$, $F_3(-D) \equiv 0 \bmod 9$ and hence $F_3(x - D)$ is by II, *B* seen to be reducible and hence $F_3(x)$ is in $k(3)$ the product of a linear and a quadratic factor. If $D^2 \not\equiv 1 \bmod 9$, $F_3(-D)$ is divisible by 3 but not by 9 and hence by II, *A*, $F_3(x - D)$ is seen to be irreducible in $k(3)$.

Hence when $D^2 \equiv 1 \bmod 9$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$ and when $D^2 \not\equiv 1 \bmod 9$, $3 \sim \mathfrak{p}^3$.

(*c*) In this case $\lambda \geqq \mu > 0$ and hence by the restrictions imposed on λ and μ we conclude that $\mu = 1$ or 2 and by II, *A*, $F_3(x)$ is irreducible in $k(3)$ and in $k(\alpha_i)$, $3 \sim \mathfrak{p}^3$.

(*d*) In this case $\lambda < \mu$ and again by the restrictions on λ and μ we see that this is possible only when $\lambda = 1$ and by II, *B* we conclude that $F_3(x)$ is in $k(3)$ the product of a linear and a quadratic factor and in $k(\alpha_i)$, $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

VII. $p = 2$.

We observe that when $p = 2$ if D is odd $s = 0$ and Δ_3 is not divisible by 2 and we need only consider $F_3(x)$ relative to the modulus 2. Two possibilities occur as follows:

$$F_3(x) \equiv x^3 + x + 1 \text{ irreducible mod } 2,$$
$$F_3(x) \equiv x^3 + 1 \equiv (x - 1)(x^2 + x + 1) \bmod 2.$$

Hence when C is odd, $F_3(x)$ is irreducible in $k(2)$ and in $k(\alpha_i)$, $2 \sim \mathfrak{p}$; and when C is even, $F_3(x)$ is in $k(2)$ the product of a linear and a quadratic factor in $k(2)$ and in $k(\alpha_i)$, $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

If $s > 0$ it is necessary that $\mu > 0$ and we shall consider the cases (*a*) $\lambda = 0$, $\mu \geqq 1$; (*b*) $\lambda \geqq \mu$; (*c*) $\lambda < \mu$.

(*a*) In this case $s \geqq 2$. We have $F_3(x) \equiv x(x^2 + C) \bmod 2^\mu$ and since $R(x, x^2 + C) = C$ is odd we conclude that in this case $F_3(x)$ has a factor in $k(2)$* and we shall write $F_3(x) = (x - a)Q(x)$. Using the same reasoning as before we see that when s is even and $\Delta_3' \equiv 1 \bmod 8$, then the quadratic factor is reducible and, in all other cases, irreducible.

We can then conclude by making use of the isomorphism previously referred to and the factorization of 2 in a quadratic domain† that when s is even, if $\Delta_3' \equiv 1 \bmod 8$, then $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$, and if $\Delta_3' \not\equiv 1 \bmod 8$ but $\Delta_3' \equiv 1 \bmod 4$, then $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$, and if $\Delta_3' \equiv 3 \bmod 4$ or when s is odd, $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

(*b*) Since in this case $\lambda \geqq \mu$ as in IV, (*c*), we can conclude that $\mu = 1$ or 2 and by II, *A*, $F_3(x)$ is irreducible in $k(2)$ and $2 \sim \mathfrak{p}^3$ in $k(\alpha_i)$.

(*c*) Since in this case $\lambda < \mu$ we know again that $\lambda = 1$ and again by II, *B* and *C*, $F_3(x)$ is in $k(2)$ the product of a linear and a quadratic factor and in $k(\alpha_i)$, $2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$.

* Hensel, A.Z., p. 71.

† Hilbert, Report, *Jahresbericht der Deut. Math. ver.*, Vol. 4.

SUMMARY.

$p > 3$

$\Delta_2^{(p-1)/2} \equiv 1 \bmod p$

$p \equiv 1 \bmod 3$

$C \equiv 0 \bmod p, \quad \Delta_3^{(p-1)/3} \equiv 1 \bmod p$
$C \not\equiv 0 \bmod p, \quad \varphi_{(p-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$ $\Big\}$ $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$

$C \equiv 0 \bmod p, \quad \Delta_3^{(p-1)/3} \not\equiv 1 \bmod p$
$C \not\equiv 0 \bmod p, \quad \varphi_{(p-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$ $\Big\}$ $p \sim \mathfrak{p}$

$p \equiv -1 \bmod 3$ ..... $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$\Delta_2^{(p-1)/2} \equiv -1 \bmod p$

$\varphi_{(p^2-1)/3}(-9D, \Delta_2) \equiv 0 \bmod p$

$p \equiv -1 \bmod 3$ ..... $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$

$p \equiv 1 \bmod 3$ ..... $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$\varphi_{(p^2-1)/3}(-9D, \Delta_2) \not\equiv 0 \bmod p$ ..... $p \sim \mathfrak{p}$

$\Delta_2 \equiv 0 \bmod p, \quad \Delta_3 = p^s \Delta_3'$

$\lambda = \mu = 0$

s even

$(\Delta_3')^{(p-1)/2} \equiv 1 \bmod p$ ..... $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$

$(\Delta_3')^{(p-1)/2} \equiv -1 \bmod p$ ..... $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

s odd ..... $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$\lambda \geqq \mu > 0$ ..... $p \sim \mathfrak{p}^3$

$0 < \lambda < \mu$ ..... $p \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$p = 3$

$\lambda = 0$

$\Delta_3 \equiv 1 \bmod 3$

$D \equiv 0 \bmod 3$ ..... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$

$D \not\equiv 0 \bmod 3$ ..... $3 \sim \mathfrak{p}$

$\Delta_3 \equiv 2 \bmod 3$ ..... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$\lambda = 1, \quad \mu = 0$

$s \geqq 7$

s even

$\Delta_3' \equiv 1 \bmod 3$ ..... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$

$\Delta_3' \equiv -1 \bmod 3$ ..... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

s odd ..... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$s = 6$

$\Delta_3' \equiv 1 \bmod 3$ ..... $3 \sim \mathfrak{p}$

$\Delta_3' \equiv -1 \bmod 3$ ..... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$s = 5$ or 4 ..... $3 \sim \mathfrak{p}^3$

$s = 3$

$D^2 + 2 \equiv 0 \bmod 9$ ..... $3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$D^2 + 2 \not\equiv 0 \bmod 9$ ..... $3 \sim \mathfrak{p}^3$

$\lambda > 1, \quad \mu = 0$

$D^2 \equiv 1 \bmod 9 \ldots\ldots 3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$D^2 \not\equiv 1 \bmod 9 \ldots\ldots 3 \sim \mathfrak{p}^3$

$\lambda \geqq \mu > 0 \ldots\ldots 3 \sim \mathfrak{p}^3$

$\lambda < \mu \ldots\ldots 3 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$p = 2$

$\mu = 0$

$C \equiv 0 \bmod 2 \ldots\ldots 2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$C \not\equiv 0 \bmod 2 \ldots\ldots 2 \sim \mathfrak{p}$

$\lambda = 0, \quad \mu \geqq 1$

s even

$\Delta_3' \equiv 1 \bmod 8 \ldots\ldots 2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3$

$\Delta_3' \not\equiv 1 \bmod 8$

$\Delta_3' \equiv 1 \bmod 4 \ldots\ldots 2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2$

$\Delta_3' \equiv 3 \bmod 4 \ldots\ldots 2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

s odd $\ldots\ldots 2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

$\lambda \geqq \mu > 0 \ldots\ldots 2 \sim \mathfrak{p}^3$

$\lambda < \mu \ldots\ldots 2 \sim \mathfrak{p}_1 \cdot \mathfrak{p}_2^2$

ON THE STRUCTURE OF FINITE CONTINUOUS GROUPS WITH A SINGLE EXCEPTIONAL INFINITESIMAL TRANSFORMATION.

By S. D. Zeldin.

Introduction.—Let $X_1, \cdots, X_r, X_{r+1}$ be the symbols of the infinitesimal transformations of a finite continuous group G with $r+1$ essential parameters, in which case

$$(X_i, X_j) = \sum_{k=1}^{r+1} c_{ijk} X_k^* \quad (i, j = 1, 2, \cdots, r+1). \tag{1}$$

The symbols of the infinitesimal transformations of the group Γ adjoint to G are then

$$D_i = \sum_{j=1}^{r+1} \sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \dagger \quad (i = 1, 2, \cdots, r+1); \tag{2}$$

and we have

$$(D_i, D_j) = \sum_{k=1}^{r+1} c_{ijk} D_k \ddagger \quad (i, j = 1, 2, \cdots, r+1). \tag{3}$$

The group Γ has $r+1$ essential parameters (i.e., there is no linear relation with constant coefficients between $D_1, \cdots, D_{r+1}$) if, and only if, G contains no exceptional§ infinitesimal transformation, i.e., no infinitesimal transformation $\sum_{i=1}^{r+1} a_i X_i$ such that

$$(\sum_{i=1}^{r+1} a_i X_i, X_j) = \sum_{i=1}^{r+1} a_i (X_i, X_j) = 0 \quad (j = 1, 2, \cdots, r+1). \tag{4}$$

Corresponding to each independent exceptional infinitesimal transformation of G, $\sum a_i X_i$, is an independent linear relation $\sum_{i=1}^{r+1} a_i D_i = 0$ between the differential operators $D_1, \cdots, D_r, D_{r+1}$.||

We shall assume in what follows that the group G has just one exceptional infinitesimal transformation, which we may take without loss of generality to be X_{r+1}. In this case,

$$\begin{aligned} (X_i, X_j) &= \sum_{k=1}^{r} c_{ijk} X_k + c_{i,\, j,\, r+1} X_{r+1} \quad (i, j = 1, 2, \cdots, r), \\ (X_i, X_{r+1}) &= (X_{r+1}, X_i) = 0 \qquad\qquad (i = 1, 2, \cdots, r, r+1); \end{aligned} \tag{5}$$

* Lie-Scheffers, Continuirliche Gruppen, p. 391.

† Lie, loc. cit., p. 466.

‡ Lie, loc. cit., p. 467.

§ Lie calls it "ausgezeichnete," loc. cit., p. 465.

|| Lie, loc. cit., p. 465.

so that

$$c_{i,\,r+1,\,k} = -\,c_{r+1,\,i,\,k} = 0 \quad (i, k = 1, 2, \cdots, r, r+1); \tag{6}$$

and, therefore,

$$D_{r+1} = 0, \qquad D_i = \sum_{j=1}^{r}\sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k}, \tag{7}$$

$$(D_i, D_j) = \sum_{k=1}^{r} c_{ijk} D_k \quad (i, j = 1, \cdots, r).$$

In consequence of the assumption that G contains just one exceptional transformation, there will exist a finite continuous group G' with r essential parameters generated by r infinitesimal transformations, whose symbols we shall denote by $Y_1, \cdots, Y_r$, such that

$$(Y_i, Y_j) = \sum_{k=1}^{r} c_{ijk} Y_k^* \qquad (i, j = 1, 2, \cdots, r). \tag{8}$$

Let Γ' denote the adjoint of G', and the symbols of its infinitesimal transformations let be $D_1', \cdots, D_r'$, where

$$D_i' \equiv \sum\sum \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r). \tag{9}$$

We have then

$$(D_i', D_j') = \sum_{k=1}^{r} c_{ijk} D_k' \qquad (i, j = 1, 2, \cdots, r). \tag{10}$$

In what follows we shall investigate the conditions that may be imposed on the structure of G' (when, as assumed, G has just one exceptional infinitesimal transformation) so that we shall have

$$c_{i,\,k,\,r+1} = -\,c_{k,\,i,\,r+1} = 0 \quad (i, k = 1, \cdots, r, r+1); \tag{11}$$

and we shall show, if there is just one spread invariant to the adjoint of G' (which will be assumed in what follows), that then

$$c_{i,\,k,\,r+1} = -\,c_{k,\,i,\,r+1} = 0 \quad (i, k = 1, \cdots, r, r+1), \tag{11}$$

or, by a suitable choice of the X's, these conditions will be satisfied. From what is stated below, it follows that, when the adjoint of G' has but one invariant spread, this spread is an $(r-1)$-spread.

Invariant spreads of the adjoint.—Lie shows† that the invariants of the adjoint of any finite continuous group may be taken homogeneous: if there is only one invariant, it will be homogeneous but not of order zero; if there are two or more invariants, all of them can be taken homogeneous and of order zero except one, which will be homogeneous but not of order zero.

* Lie-Engel, "Transformations Gruppen," Vol. I, pp. 300–305.

† Lie-Scheffers, "Continuirliche Gruppen," pp. 596–599.

Every function of $\alpha_1, \cdots, \alpha_r$ invariant to the adjoint Γ' of G', therefore, yields a spread invariant to Γ'. Since, by supposition, the adjoint of G' has but one invariant spread, there is therefore but one function of $\alpha_1, \cdots, \alpha_r$ invariant to the adjoint of G'. This function will be denoted, in this paper, by $\varphi \equiv \varphi(\alpha_1, \cdots, \alpha_r)$, homogeneous in the α's; and then $\varphi(\alpha) = 0$ will be the only spread invariant to the adjoint of G'.

The function $\varphi(\alpha)$ is a solution of the system of equations

$$D_i' f(\alpha) \equiv \sum_{j=1}^{r} \alpha_j c_{ji1} \frac{\partial f(\alpha)}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{jir} \frac{\partial f(\alpha)}{\partial \alpha_r} = 0 \quad (i = 1, 2, \cdots, r). \tag{12}$$

From $\partial\varphi(\alpha)/\partial\alpha_{r+1} = 0$ it follows, by (7), that

$$D_i\varphi(\alpha) \equiv \sum_{j=1}^{r} \alpha_j c_{ji1} \frac{\partial \varphi(\alpha)}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{jir} \frac{\partial \varphi(\alpha)}{\partial \alpha_r} \equiv D_i'\varphi(\alpha) = 0 \quad (i = 1, 2, \cdots, r); \tag{13}$$

therefore, $\varphi(\alpha)$ is also an invariant of the adjoint of G.

It is convenient to denote by

$$E_i \equiv \begin{pmatrix} c_{i11}, & \cdots, & c_{ir1}, & 0 \\ c_{i12}, & \cdots, & c_{ir2}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{i1r}, & \cdots, & c_{irr}, & 0 \\ c_{i,\,1,\,r+1}, & \cdots, & c_{i,\,r,\,r+1}, & 0 \end{pmatrix} \quad (i = 1, 2, \cdots, r) \tag{14}$$

the matrix of the differential operators

$$D_i \equiv \sum_{j=1}^{r} \sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r).$$

Then

$$\sum a_i E_i \equiv \begin{pmatrix} \sum a_i c_{i,\,1,\,1}, & \cdots, & \sum a_i c_{i,\,r,\,1}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sum a_i c_{i,\,1,\,r}, & \cdots, & \sum a_i c_{i,\,r,\,r}, & 0 \\ \sum a_i c_{i,\,1,\,r+1}, & \cdots, & \sum a_i c_{i,\,r,\,r+1}, & 0 \end{pmatrix} \tag{15}$$

is the matrix of the general infinitesimal transformation $\sum_{i=1}^{r} a_i D_i$ of Γ. Similarly we shall write

$$\mathcal{E}_i \equiv \begin{pmatrix} c_{i11}, & \cdots, & c_{ir1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{i1r}, & \cdots, & c_{irr} \end{pmatrix} \qquad (i = 1, 2, \cdots, r) \tag{16}$$

to denote the matrix of the differential operators

$$D_i' \equiv \sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r);$$

and then

$$\sum_{1}^{r} a_i \mathcal{E}_i \equiv \begin{vmatrix} \sum a_i c_{i11}, & \cdots, & \sum a_i c_{ir1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \sum a_i c_{i1r}, & \cdots, & \sum a_i c_{irr} \end{vmatrix} \tag{17}$$

will denote the matrix of the general infinitesimal transformation $\sum_{i=1}^{r} a_i D_i'$ of Γ'.

Any invariant of the adjoint of G being a solution of the complete system of equations

$$D_i f(\alpha) \equiv \sum_{j=1}^{r+1}\sum_{k=1} \alpha_j c_{jik} \frac{\partial f(\alpha)}{\partial \alpha_k} = 0 \qquad (i = 1, 2, \cdots, r), \tag{18}$$

the number of invariants of the group Γ, adjoint to G, is determined by the array

$$\begin{matrix} \sum_{1}^{r} \alpha_j c_{j11}, & \cdots, & \sum_{1}^{r} \alpha_j c_{j1r}, & \sum \alpha_j c_{j,\,1,\,r+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sum \alpha_j c_{jr1}, & \cdots, & \sum \alpha_j c_{jrr}, & \sum \alpha_j c_{j,\,r,\,r+1} \end{matrix} \tag{19}$$

being equal to $r + 1$ less the order of the non-zero determinant of highest order formed from the array, and thus to the nullity of $\sum \alpha_j \breve{E}_j$, where $\breve{E}_j$ denotes the transverse of E_j. Therefore, the number of invariants of G is equal to the nullity of $\sum \alpha_j E_j$, being a solution of the complete system of equations

$$D_i' f(\alpha) \equiv \sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial f(\alpha)}{\partial \alpha_k} = 0 \qquad (i = 1, 2, \cdots, r). \tag{20}$$

The number of invariants of the adjoint of G' is equal to the nullity of the array

$$\begin{matrix} \sum_{1}^{r} \alpha_j c_{j11}, & \cdots, & \sum_{1}^{r} \alpha_j c_{j1r} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \sum_{1}^{r} \alpha_j c_{jr1}, & \cdots, & \sum_{1}^{r} \alpha_j c_{jrr} \end{matrix} \tag{21}$$

and thus is equal to the nullity of $\sum_1^r \alpha_j \breve{\mathcal{E}}_j$ (where, as before, $\breve{\mathcal{E}}_j$ is the transverse of $\mathcal{E}_j$), or, what is the same thing, to the nullity of $\sum_1^r \alpha_j \mathcal{E}_j$.

Since it is assumed that the adjoint of G' has just one invariant, it follows that the nullity of $\sum_1^r \alpha_i \mathcal{E}_i$ is one for an arbitrary system of values

of the α's, i.e., at least one of the minors of $|\sum\alpha_i\mathcal{E}_i|$* of order $r-1$ is not zero. But every minor of $|\sum\alpha_i\mathcal{E}_i|$ is a minor of $|\sum\alpha_iE_i|$, therefore at least one minor of the order $r-1$ of the determinant $|\sum\alpha_iE_i|$ is not zero, and thus the nullity of the matrix $\sum\alpha_iE_i$ *cannot exceed two* for an arbitrary system of values of the α's.

Further, for $\alpha_1, \cdots, \alpha_r, \alpha_{r+1}$ assigned, the symbolic equation

$$\left(\sum_1^{r+1}\alpha_iX_i,\ \sum_1^{r+1}\eta_iX_i\right)=0, \tag{22}$$

is satisfied for

$$\eta_1=\alpha_1, \cdots, \eta_r=\alpha_r, \eta_{r+1}=0 \quad \text{and} \quad \eta_1=0, \cdots, \eta_r=0, \eta_{r+1}=1.$$

But from (22) follows

$$\sum\alpha_iE_i(\eta_1, \cdots, \eta_{r+1}) = \begin{vmatrix} \sum\alpha_ic_{i11}, & \cdots \\ \sum\alpha_ic_{i,\,1,\,r+1}, & \cdots \end{vmatrix} (\eta_1, \cdots, \eta_{r+1}) = 0.\dagger \tag{23}$$

Therefore, this system of linear equations has two independent solutions, namely,

$$\eta_1=\alpha_1, \cdots, \eta_r=\alpha_r, \eta_{r+1}=0 \quad \text{and} \quad \eta_1=0, \cdots, \eta_r=0, \eta_{r+1}=1.$$

Whence it follows that the nullity of the matrix $\sum\alpha_iE_i$, for any system of values of the α's, is at least two. Wherefore, the nullity of $\sum\alpha_iE_i$, for an arbitrary system of values of the α's, is just *equal to two;* and thus the number of *invariants* of the adjoint of G is *equal to two.*

The Holomorphic Invariant of the Adjoint.—Let the second invariant of the adjoint of G, independent of $\varphi(\alpha)$, be $\psi\equiv\psi(\alpha, \cdots, \alpha_{r+1})$ homogeneous in the α's. We must have

$$\frac{\partial\psi(\alpha)}{\partial\alpha_{r+1}}\not\equiv 0. \tag{24}$$

For, otherwise, if $\dfrac{\partial\psi(\alpha)}{\partial\alpha_{r+1}}\equiv 0$, we get

$$D_i'\psi(\alpha)\equiv D_i\psi(\alpha)\equiv 0, \tag{25}$$

and then, since the group adjoint to G' has but one invariant, we should have $\psi=W(\varphi)$, which is contrary to our assumption.

By Lie's theorem, as stated above, $\psi(\alpha)$ can be taken homogeneous (and of order zero, since a homogeneous invariant of the adjoint of G, viz., $\varphi(\alpha_1, \cdots, \alpha_r)$, not of order zero, exists); and thus, we may put

$$\psi(\alpha)\equiv F\left(\frac{\alpha_1}{\alpha_{r+1}}, \cdots, \frac{\alpha_r}{\alpha_{r+1}}\right). \tag{26}$$

* The determinant of any matrix M will be denoted by $|M|$.

† Lie-Scheffers, "Continuirliche Gruppen," pp. 558, 562. The notation used here is due to Cayley, *Philosophical Transactions*, 1858.

Let

$$\beta_1 = \frac{\alpha_1}{\alpha_{r+1}}, \cdots, \beta_r = \frac{\alpha_r}{\alpha_{r+1}}, \beta_{r+1} = \alpha_{r+1}; \tag{27}$$

then

$$\alpha_1 = \beta_1 \beta_{r+1}, \cdots, \alpha_r = \beta_r \beta_{r+1}, \alpha_{r+1} = \beta_{r+1} \tag{28}$$

and

$$\frac{\partial F(\beta)}{\partial \beta_{r+1}} \equiv 0. \tag{29}$$

Let

$$\begin{aligned} \overline{D}_i &\equiv D_i\beta_1 \frac{\partial}{\partial \beta_1} + \cdots + D_i\beta_r \frac{\partial}{\partial \beta_r} + D_i\beta_{r+1} \frac{\partial}{\partial \beta_{r+1}} \\ &\equiv \left(\sum_{j=1}^{r} c_{ji1}\beta_j - \beta_1 \sum_{j=1}^{r} c_{jir+1} \right) \frac{\partial}{\partial \beta_1} \\ &\quad + \cdots + \left(\sum_{j=1}^{r} c_{jir}\beta_j - \beta_r \sum_{1}^{r} c_{jir+1}\beta_j \right) \frac{\partial}{\partial \beta_r} \\ &\quad + \beta_{r+1} \sum_{1}^{r} c_{jir+1}\beta_j \frac{\partial}{\partial \beta_{r+1}} \quad (i = 1, 2, \cdots, r). \end{aligned} \tag{30}$$

Then, if $f(\alpha) \equiv ff(\beta)$,

$$\overline{D}_i ff(\beta) = D_i f(\alpha) \qquad (i = 1, 2, \cdots, r). \tag{31}$$

Therefore,

$$\begin{aligned} &\left(\sum_{1}^{r} c_{ji1}\beta_j - \beta_1 \sum_{1}^{r} c_{jir+1}\beta_j \right) \frac{\partial F(\beta)}{\partial \beta_1} + \cdots + \beta_{r+1} \sum_{1}^{r} c_{jir+1}\beta_j \frac{\partial F(\beta)}{\partial \beta_{r+1}} \\ &\qquad \equiv \overline{D}_i F(\beta) \equiv D_i \psi(\alpha) \equiv 0 \quad (i = 1, 2, \cdots, r). \end{aligned} \tag{32}$$

It follows that $F(\beta)$ is a solution of the system of equations

$$\overline{D}_1 f(\beta) = 0, \cdots, \overline{D}_r f(\beta) = 0. \tag{33}$$

Since the coefficients of this system of equations are rational integral functions of the β's, the solution $F(\beta)$ of this system may be taken holomorphic in the neighborhood of $\beta_1 = 0, \cdots, \beta_r = 0, \beta_{r+1} = 1$. Consequently

$$\psi(\alpha) \equiv F\left(\frac{\alpha_1}{\alpha_{r+1}}, \cdots, \frac{\alpha_r}{\alpha_{r+1}} \right)$$

may be taken holomorphic in the neighborhood of $\alpha_1 = 0, \cdots, \alpha_r = 0$, $\alpha_{r+1} = 1$. There are two cases: first, $\psi(\alpha)$ is algebraic; second, $\psi(\alpha)$ is transcendental.

The Function $\psi(\alpha)$ Algebraic Invariant.—The invariant $\psi(\alpha)$ is now assumed to be algebraic. In this case the invariant spread $\psi(\alpha) = 0$ is represented by the rational integral equation of degree m homogeneous in the α's,

$$\Psi(\alpha) \equiv u_0(\alpha)\alpha_{r+1}^m + u_1(\alpha)\alpha_{r+1}^{m-1} + \cdots + u_k(\alpha)\alpha_{r+1}^{m-k} + \cdots + u_m(\alpha) = 0, \tag{34}$$

where $u_k(\alpha)$ is a polynomial of degree k homogeneous in $\alpha_1, \cdots, \alpha_r$.

Since the point $(0, \cdots, 0, 1)$, i.e., the point $\alpha_1 = \cdots = \alpha_r = 0, \alpha_{r+1} = 1$, corresponding to the exceptional infinitesimal transformation X_{r+1} of G, is invariant to the adjoint of G,* the polars of $(0, \cdots, 0, 1)$ quâ $\Psi(\alpha) = 0$ are all invariant to the adjoint of G. These polars are:

$$\begin{aligned}
\frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} &\equiv \frac{m!}{(m-1)!}u_0(\alpha)\alpha_{r+1}^{m-1} + \frac{(m-1)!}{(m-2)!}u_1(\alpha)\alpha_{r+1}^{m-2} \\
&\qquad + \cdots + \frac{1!}{0!}u_{m-1}(\alpha) = 0, \\
\frac{\partial^2\Psi(\alpha)}{\partial\alpha_{r+1}^2} &\equiv \frac{m!}{(m-2)!}u_0(\alpha)\alpha_{r+1}^{m-2} + \frac{(m-1)!}{(m-3)!}u_1(\alpha)\alpha_{r+1}^{m-3} \\
&\qquad + \cdots + \frac{2!}{0!}u_{m-2}(\alpha) = 0, \\
&\cdots\cdots\cdots

where

$$\alpha_i^{(r+1)} = 0 \quad (i \neq r+1), \qquad \alpha_{r+1}^{(r+1)} = 1.$$

In Cayley's notation this transformation can be written:

$$(\alpha_1, \cdots, \alpha_r, \alpha_{r+1}) = \begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_1^{(r)}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{r+1}^{(1)}, & \cdots, & \alpha_{r+1}^{(r)}, & 0 \end{vmatrix} (\bar{\alpha}_1, \cdots, \bar{\alpha}_r, \bar{\alpha}_{r+1}).$$

And now if

$$\bar{X}_i = \alpha_i^{(1)} X_1 + \cdots + \alpha_r^{(1)} X_r + \alpha_{r+1}^{(i)} X_{r+1} \quad (i = 1, 2, \cdots, r, r+1), \tag{38}$$

or

$$(\bar{X}_1, \cdots, \bar{X}_r, \bar{X}_{r+1}) = \begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)}, & \alpha_{r+1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)}, & \alpha_{r+1}^{(r)} \\ 0, & \cdots, & 0, & 1 \end{vmatrix} (X_1, \cdots, X_r, X_{r+1}),$$

then

$$\sum_{i=1}^{r+1} \bar{\alpha}_i \bar{X}_i = \sum_{i=1}^{r+1} \alpha_i X_i. \tag{39}$$

The infinitesimal transformations $\bar{X}_1, \cdots, \bar{X}_r, \bar{X}_{r+1}$ will then be independent, since the determinant

$$\begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)}, & \alpha_{r+1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)}, & \alpha_{r+1}^{(r)} \\ 0, & \cdots, & 0, & 1 \end{vmatrix} = \begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)} \end{vmatrix} = 0;$$

and we shall have

$$(\bar{X}_i, \bar{X}_j) = \sum_{k=1}^{r+1} \bar{c}_{ijk} \bar{X}_k \quad (i, j = 1, 2, \cdots, r, r+1). \tag{40}$$

Since $\bar{X}_1, \cdots, \bar{X}_r$ are represented by the points $\alpha^{(1)}, \cdots, \alpha^{(r)}$ in the $(r-1)$-flat

$$\frac{(m-1)!}{1!} u_0(\alpha)\alpha_{r+1} + \frac{(m-1)!}{0!} u_1(\alpha) = 0$$

which is invariant to the adjoint of G, they will constitute an invariant subgroup of G;* and thus

$$(\bar{X}_i, \bar{X}_j) = \sum_{k=1}^{r} c_{ijk} \bar{X}_k \quad (i = 1, 2, \cdots, r; j = 1, \cdots, r, r+1). \tag{41}$$

* Lie-Scheffers, "Continuirliche Gruppen," pp. 485–487.

Further, $\overline{X}_{r+1} = X_{r+1}$. Whence

$$\bar{c}_{ijr+1} = -\bar{c}_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r+1), \tag{42}$$

which was to be proved. We may therefore assume that $u_0(\alpha) = 0$.

If now $m = 1$, the equation of the spread invariant to the adjoint of G is $\Psi(\alpha) \equiv u_1(\alpha) = 0$. But in this case $\dfrac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} \equiv 0$, which is contrary to our assumption.

Let $m = 2$. The equations of the invariant spread and the invariant polars are then

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} + u_2(\alpha) = 0, \qquad \frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} \equiv u_1(\alpha) = 0.$$

Let

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \not\equiv 0, \qquad u_2(\alpha) \not\equiv u_1(\alpha)\cdot v_1(\alpha),$$

where $v_1(\alpha)$ is a function linear in $\alpha_1, \cdots, \alpha_r$. If now

$$u_1(\alpha) = 0, \qquad \text{then} \qquad \frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} = 0;$$

and since this equation is invariant, $u_1(\alpha') = \dfrac{\partial\Psi(\alpha')}{\partial\alpha'_{r+1}} = 0$, where $\alpha'_1, \cdots, \alpha'_r$, α'_{r+1} are the coördinates of the point obtained by applying the infinitesimal transformations of the adjoint of G to the point $(\alpha_1, \cdots, \alpha_{r+1})$. Again, if

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0,$$

then

$$\frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} \equiv u_1(\alpha) = 0, \qquad \Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} + u_2(\alpha) = 0;$$

and, since these equations are separately invariant to the adjoint of G, it follows that

$$\frac{\partial\Psi(\alpha')}{\partial\alpha'_{r+1}} \equiv u_1(\alpha') = 0, \qquad \Psi(\alpha') \equiv u_1(\alpha')\alpha'_{r+1} + u_2(\alpha') = 0;$$

and thus

$$u_1(\alpha') = 0, \qquad u_2(\alpha') = 0.$$

Therefore, the spread

$$u_1(\alpha) = 0$$

and the spread

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0$$

are both invariant to the adjoint of G', which is contrary to the assumption that the adjoint of G' has but one invariant spread.

If

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \equiv 0,$$

then the invariant spread

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} = 0$$

is reducible. In this case each component of that spread is invariant to the adjoint of G; and therefore,

$$\alpha_{r+1} = 0$$

is an invariant flat not passing through the point $(0, \cdots, 0, 1)$, a case already treated above. In this case we can clearly see that

$$c_{ijr+1} = -\, c_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r+1).$$

If

$$u_1(\alpha) \equiv 0, \qquad u_2(\alpha) \not\equiv 0,$$

then

$$\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv 0,$$

which is contrary to the assumption.

If

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \equiv v_1(\alpha) \cdot u_1(\alpha) \not\equiv 0,$$

where $v_1(\alpha)$ denotes a function linear in $\alpha_1, \cdots, \alpha_r$, then the invariant spread

$$\Psi(\alpha) \equiv u_1(\alpha) \cdot \alpha_{r+1} + u_2(\alpha) \equiv u_1(\alpha)[\alpha_{r+1} + v_1(\alpha)] = 0$$

is reducible; and the component flat

$$\alpha_{r+1} + v_1(\alpha) = 0$$

is invariant and does not pass through the point $(0, \cdots, 0, 1)$, in which case, as was shown above, we can, by a suitable choice of $X_1, \cdots, X_r, X_{r+1}$, make

$$c_{ijr+1} = -\, c_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r+1).$$

Let now m be any positive integer greater than 2. The equations of the invariant spread and the invariant polars are then

$$\begin{aligned}
\Psi(\alpha) &\equiv u_1(\alpha)\alpha_{r+1}^{m-1} + u_2(\alpha)\alpha_{r+1}^{m-2} + \cdots + u_m(\alpha) = 0, \\
\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} &\equiv \frac{(m-1)!}{(m-2)!} u_1(\alpha) \cdot \alpha_{r+1}^{m-2} + \cdots + \frac{1!}{0!} u_{m-1}(\alpha) = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial^{m-2} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-2}} &\equiv \frac{(m-1)!}{1!} u_1(\alpha)\alpha_{r+1} + \frac{(m-2)!}{0!} u_2(\alpha) = 0, \\
\frac{\partial^{m-1} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-1}} &\equiv \frac{(m-1)!}{0!} u_1(\alpha) = 0.
\end{aligned}$$

First, let $u_1(\alpha) \not\equiv 0$. Then either $u_2(\alpha) \not\equiv 0$ does not contain $u_1(\alpha)$, or

$u_2(\alpha)$, $u_3(\alpha)$, $\cdots$, $u_p(\alpha)$ $(p \leqq m)$ each contain $u_1(\alpha)$. In the former case, as we have seen before, the spread

$$u_1(\alpha) = 0$$

and the spread

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0$$

are both invariant to the adjoint of G', which is contrary to our assumption. In the latter case

$$\begin{aligned} u_2(\alpha) &= v_1(\alpha) \cdot u_1(\alpha), \\ u_3(\alpha) &= v_2(\alpha) \cdot u_1(\alpha), \\ &\cdots \\ u_{p-1}(\alpha) &= v_{p-2}(\alpha) \cdot u_1(\alpha), \\ u_p(\alpha) &= v_{p-1}(\alpha) \cdot u_1(\alpha), \end{aligned}$$

where $v_k(\alpha)$ is a function of degree k homogeneous in $\alpha_1, \cdots, \alpha_r$. In this case each of the invariant spreads

$$\frac{\partial^{m-q}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-q}} = 0 \qquad (q = 2, 3, \cdots, p)$$

is reducible. Therefore, the component flat

$$\frac{(m-1)!}{1!}\alpha_{r+1} + \frac{(m-2)!}{0!}v_1(\alpha) = 0$$

is invariant. This case has been treated above.

Secondly, let

$$u_1(\alpha) \equiv 0, \qquad u_2(\alpha) \not\equiv 0.$$

Either $u_3(\alpha) \not\equiv 0$ does not contain $u_2(\alpha)$, or $u_3(\alpha)$, $u_4(\alpha)$, $\cdots$, $u_p(\alpha)$ $(p \leqq m)$ each contain $u_2(\alpha)$. In the former case, by reasoning similar to that employed before, it appears that the spread

$$u_2(\alpha) = 0$$

and the spread

$$u_2(\alpha) = 0, \qquad u_3(\alpha) = 0$$

are both invariant to the adjoint of G'. In the latter case

$$\begin{aligned} u_3(\alpha) &= v_1(\alpha) \cdot u_2(\alpha), \\ &\cdots \\ u_{p-1}(\alpha) &= v_{p-3}(\alpha) \cdot u_2(\alpha), \\ u_p(\alpha) &= v_{p-2}(\alpha) \cdot u_2(\alpha), \end{aligned}$$

where $v_k(\alpha)$ is a function of order k homogeneous in $\alpha_1, \cdots, \alpha_r$. In this case each of the invariant spreads

$$\frac{\partial^{m-q}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-q}} = 0 \qquad (q = 3, 4, \cdots, p)$$

is reducible. In particular the invariant spread

$$\frac{\partial^{m-3}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-3}} \equiv \frac{(m-2)!}{2!}u_2(\alpha)\cdot\alpha_{r+1} + \frac{(m-3)!}{1!}u_3(\alpha)$$
$$\equiv u_2(\alpha)\left[\frac{(m-2)!}{2!}\alpha_{r+1} + \frac{(m-3)!}{1!}v_1(\alpha)\right] = 0$$

is reducible. Therefore, the component flat

$$\frac{(m-2)!}{2!}\alpha_{r+1} + \frac{(m-3)!}{1!}v_1(\alpha) = 0$$

is invariant. This case has also been treated above.

By the same reasoning we may treat the other similar cases which arise.

The Function $\psi(\alpha)$ Transcendental Invariant.—Since the point $\alpha^{(1)} = (0, \cdots, 0, 1)$ is invariant to the adjoint of G, the spreads

$$\sum_{i=1}^{r+1}\alpha_i\left[\frac{\partial\psi(\alpha)}{\partial\alpha_i}\right]_{\alpha=\alpha^{(1)}} = 0,$$
$$\sum_{i=1}^{r+1}\sum_{j=1}^{r+1}\alpha_i\alpha_j\left[\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j}\right]_{\alpha=\alpha^{(1)}} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{i_1=1}^{r+1}\cdots\sum_{i_\mu=1}^{r+1}\alpha_{i_1}\cdots\alpha_{i_\mu}\left[\frac{\partial^\mu\psi(\alpha)}{\partial\alpha_{i_1}\cdots\partial\alpha_{i_\mu}}\right]_{\alpha=\alpha^{(1)}} = 0,$$
etc. $\cdots\cdots\cdots\cdots\cdots$

are invariant to the adjoint of G. Wherefore, there is an algebraic spread invariant to the adjoint of G, a case already treated above, or all the partial differential coefficients of $\psi(\alpha)$ quâ $\alpha_1, \cdots, \alpha_{r+1}$ are zero for $\alpha = \alpha^{(1)}$. But this is impossible as is shown below.

We have seen that

$$\psi(\alpha) = F(\beta_1, \cdots, \beta_r),$$

where

$$\beta_1 = \frac{\alpha_1}{\alpha_{r+1}}, \quad \cdots, \quad \beta_r = \frac{\alpha_r}{\alpha_{r+1}}, \quad \beta_{r+1} = \alpha_{r+1},$$

can be taken analytic in the neighborhood of the point $\beta_1^{(1)} = \cdots = \beta_r^{(1)} = 0$, $\beta_{r+1}^{(1)} = 1$; and, therefore,

$$\psi(\alpha) = F(\beta) = F(\beta^{(1)}) + \sum_{i=1}^{r}\beta_i\left[\frac{\partial F(\beta)}{\partial\beta_i}\right]_{\beta=\beta^{(1)}}$$
$$+\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\beta_i\beta_j\left[\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_j}\right] + \cdots + \cdots.$$

Since

$$\frac{\partial\psi(\alpha)}{\partial\alpha_i} = \frac{\partial F(\beta)}{\partial\beta_i}\cdot\frac{1}{\alpha_{r+1}} \qquad (i = 1, 2, \cdots, r),$$
$$\frac{\partial\psi(\alpha)}{\partial\alpha_{r+1}} = -\frac{\alpha_1}{\alpha_{r+1}^2}\cdot\frac{\partial F(\beta)}{\partial\beta_1} - \cdots - \frac{\alpha_r}{\alpha_{r+1}^2}\cdot\frac{\partial F(\beta)}{\partial\beta_r},$$

we should have, if

$$\left[\frac{\partial\psi(\alpha)}{\partial\alpha_i}\right]_{\alpha=\alpha^{(1)}} = 0 \qquad (i = 1, 2, \cdots, r, r+1),$$
$$\left[\frac{\partial F(\beta)}{\partial\beta_i}\right]_{\beta=\beta^{(1)}} = 0 \qquad (i = 1, 2, \cdots, r).$$

Since

$$\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j} = \frac{1}{\alpha_{r+1}^2}\cdot\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_j} \qquad (i = 1, 2, \cdots, r),$$
$$\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_{r+1}} = \frac{\partial^2\psi(\alpha)}{\partial\alpha_{r+1}\partial\alpha_i} = \frac{1}{\alpha_{r+1}^2}\cdot\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_{r+1}} \qquad (i = 1, 2, \cdots, r+1),$$

we should have, if

$$\left[\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j}\right]_{\alpha=\alpha^{(1)}} = 0 \quad (i = 1, 2, \cdots, r+1),$$
$$\left[\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_j}\right]_{\beta=\beta^{(1)}} = 0 \quad (i = 1, 2, \cdots, r), \text{ etc. } \cdots.$$

Thus in case the partial differential coefficients of $\psi(\alpha)$ quâ $\alpha_1, \cdots, \alpha_{r+1}$ are all zero for $\alpha = \alpha^{(1)}$, then

$$\psi(\alpha) \equiv F(\beta) \equiv 0,$$

which is contrary to the assumption.

Note.—Not in all the groups with $r+1$ essential parameters and one exceptional infinitesimal transformations can all the c_{ijr+1}'s $(i, j = 1, 2, \cdots, r)$ be made zero. The group with 5 essential parameters and one exceptional infinitesimal transformation, isomorphic (not holoedrically) with the integrable group of 4 essential parameters of the type,

$$(Y_1, Y_2) = 0, \quad (Y_1, Y_3) = 0, \quad (Y_2, Y_3) = Y_1,$$
$$(Y_1, Y_4) = Y_1, \quad (Y_2, Y_4) = Y_2, \quad (Y_3, Y_4) = 0,^*$$

may serve as an illustration where not all the c_{ijr}'s $(i, j = 1, 2, 3, 4)$ can be made zero.

CLARK UNIVERSITY,
June 14, 1917.

* Lie-Scheffers, "Continuirliche Gruppen," p. 582.

THE LAPLACE-POISSON MIXED EQUATION.

By K. P. Williams.

In a recent number of this Journal * Borden discussed the Laplace-Poisson Mixed Equation

$$f'(x+1) + p(x)f'(x) + q(x)f(x+1) + r(x)f(x) = 0. \tag{1}$$

He obtains two invariants that form a fundamental system for the equation. These invariants are

$$I(x) = \frac{r(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)},$$

$$J(x) = \frac{r(x)}{p(x)} - q(x-1).$$

If either invariant is zero, the equation takes a simple form. For instance, if $I(x) \equiv 0$, i.e., if

$$r(x) \equiv p'(x) + p(x)q(x), \tag{2}$$

the equation (1) reduces to

$$f(x+1) + p(x)f(x) \equiv Ce^{-\int q(x)dx}, \tag{3}$$

where C is an arbitrary constant. This is a linear non-homogeneous difference equation of the first order. Borden obtains a solution of it by means of the symbolic solution $\Sigma G(x)$ of the equation

$$F(x+1) - F(x) = G(x),$$

where $G(x)$ is a known function.

Solutions obtained in this way are purely formal, and may have no real significance. Borden assumes at the outset of his paper that $p(x)$, $q(x)$, and $r(x)$ are analytic functions, but he nowhere makes an investigation to determine whether this hypothesis is sufficient to bestow any validity upon his results.

It is the purpose of this paper to investigate in certain cases the analytic character of the solutions Borden obtains. To do this it is necessary to make use of the existence theorems for linear difference equations. We shall state here the one that is sufficient for several cases that arise.

* R. F. Borden, "On the Laplace-Poisson Mixed Equation," American Journal of Mathematics, Vol. XLII, 1920, pp. 257–277.

THEOREM A.* *Consider the linear non-homogeneous difference equation*

$$g(x+1) + a(x)g(x) = b(x), \tag{4}$$

where $a(x)$ and $b(x)$ are rational functions, so that

$$a(x) = x^{\mu}\left(a_0 + \frac{a_1}{x} + \cdots\right),$$

$$b(x) = x^{\nu}\left(b_0 + \frac{b_1}{x} + \cdots\right), \qquad |x| > R.$$

There exists a series $g(x)$, of the form

$$\bar{g}(x) = x^{\nu-\mu}\left(g_0 + \frac{g_1}{x} + \cdots\right), \qquad \text{if} \qquad \mu > 0,$$

$$\bar{g}(x) = x^{\nu}\left(g_0 + \frac{g_1}{x} + \cdots\right), \qquad \text{if} \qquad \mu \leqq 0,$$

which formally satisfies (4), *but which in general diverges.*

There are two solutions $g_1(x)$ and $g_2(x)$ with the following properties. The function $g_1(x)$ is analytic save at zeros of $a(x)$, poles of $b(x)$, and points congruent (mod 1) to these points on the left; and it is asymptotic to the formal series $\bar{g}(x)$ in the right half plane. The function $g_2(x)$ is analytic save for poles at the poles of $a(x)$, $b(x)$, and the points congruent to these points on the right; and it is asymptotic to $\bar{g}(x)$ in the left half plane.†

From the relation (2) it is possible to determine any one of the three functions p, q, r in terms of the other two. We shall assume that p and q have the form of rational functions at infinity, so that

$$p(x) = x^{m}\left(p_0 + \frac{p_1}{x} + \cdots\right),$$

$$q(x) = x^{n}\left(q_0 + \frac{q_1}{x} + \cdots\right), \qquad |x| > R. \tag{5}$$

There are four cases to consider, according as $n < -1$, $n = -1$, $n = 0$, $n > 0$.

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* K. P. Williams, "The Solutions of Non-Homogeneous Linear Difference Equations and their Asymptotic Form," *Transactions of the American Math. Soc.*, Vol. XIV (1913), pp. 209–240.

† In order to be assured of solutions and to know their asymptotic forms, it is sufficient to assume that $a(x)$ and $b(x)$ have the form of rational functions at infinity. In that case, nothing can be said of the nature or situation of the singularities of the solutions in the finite part of the plane.

§ 1.

Case 1. Let $n < -1$; then we have

$$e^{-\int q(x)dx} = \left(1 + \frac{}{x^{-n-1}} + \frac{}{x^{-n-2}} + \cdots\right).^{*}$$

It is seen that equation (3) is now in the form (4), with $\mu = m$, $\nu = 0$. It follows from Theorem A that equation (3), and therefore equation (1), has two solutions, analytic in general, and these solutions are asymptotic in the right and left half planes, respectively, to a series

$$\bar{f}(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right), \qquad \text{if} \qquad m > 0,$$

$$\bar{f}(x) = f_0 + \frac{f_1}{x} + \cdots, \qquad \text{if} \qquad m \leqq 0,$$

which can be determined by direct substitution in (3).

The formal solution $\bar{f}(x)$ can, however, be directly obtained from (1). The calculation will be made only for the case $m > 0$.

When we substitute in (2) the values of $p(x)$ and $q(x)$ given in (5), we obtain

$$r(x) = x^{m-1}\left(mp_0 + \frac{}{x} + \cdots\right),$$

the highest power of x being the $(m-1)$th, since the assumption $n < -1$ gives $m - 1 > m + n$. The equation (1) then takes the form

$$f'(x+1) + x^m\left(p_0 + \frac{p_1}{x} + \cdots\right)f'(x)$$
$$+ x^n\left(q_0 + \frac{q_1}{x} + \cdots\right)f(x+1) + x^{m-1}\left(mp_0 + \frac{}{x} + \cdots\right)f(x) = 0.$$

Assume

$$f(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

and substitute, giving

$$-\frac{1}{x^{m+1}}(mf_0 + \cdots) - \frac{1}{x}(p_0 + \cdots)(mf_0 + \cdots)$$
$$+ \frac{1}{x^{m-n}}(q_0 + \cdots)(f_0 + \cdots) + \frac{1}{x}(mp_0 + \cdots)(f_0 + \cdots) = 0,$$

where we have omitted only powers of $1/x$. Remembering that $m > 0$, $n < -1$, we see that $f_0, f_1, f_2, \cdots$ can be determined step by step, the quantity f_0 being arbitrary.†

* We shall omit writing coefficients in series in $1/x$ where the coefficients can be determined in terms of given quantities, and their explicit form is of no concern.

† If $m \leqq 0$ it is found that f_0 is arbitrary, while some of the next coefficients, $f_1, f_2, \ldots$, are zero, the exact number having that value depending on m and n.

THEOREM 1. *Assume the coefficients p, q, and r satisfy the following hypotheses*:

(1) $$r(x) \equiv p'(x) + p(x)q(x).$$

(2) *$p(x)$ has a pole of order m at infinity ($m = 0$, if $p(x)$ is analytic at infinity).*

(3) *$q(x)$ is analytic at infinity, and has a zero there of at least the second order.*

Then there is a formal solution $\bar{f}(x)$ of equation (1) *of the form*

$$\bar{f}(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

which can be determined by direct substitution (f_0 is arbitrary). There are two solutions of (1), *namely $f_1(x)$ and $f_2(x)$, analytic in general in the finite plane. Furthermore, $f_1(x)$ is asymptotic to $\bar{f}(x)$ in the right half plane, and $f_2(x)$ is asymptotic to $\bar{f}(x)$ in the left half plane.*

§ 2.

Case 2. Let $n = -1$. In this case

$$e^{-\int q(x)dx} = x^{-q_0}\left(1 + \frac{}{x} + \frac{}{x^2} + \cdots\right).$$

The equation (3) is therefore of the form (4) with $\nu = -q_0$.*

By application of Theorem A we deduce

THEOREM 2. *Let p, q, and r satisfy hypotheses* (1) *and* (2) *of Theorem* 1, *and in addition suppose*

(3) *$q(x)$ is analytic at infinity, with a zero of the first order, and* $\lim_{x=\infty} xf(x) = q_0$.

Then there is a formal solution $\bar{f}(x)$ of (1) *of the form*

$$\bar{f}(x) = x^{-q_0-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

which can be determined by substitution. There are two solutions $f_1(x)$ and $f_2(x)$ with properties similar to those described in Theorem 1.

§ 3.

Case 3. Let $n = 0$. In this case

$$e^{-\int q(x)dx} = e^{-q_0 x}x^{-q_1}\left(1 + \frac{}{x} + \cdots\right),$$

* It is not necessary that ν be an integer in (3). In fact the substitution

$$g(x) = x^{\nu}G(x)$$

will reduce (4) to a normal form with $\nu = 0$, and in which the expansion of $a(x)$ still begins with x^{μ}.

so that equation (3) is not of the form (4). It can, however, be reduced to that form by the substitution

$$f(x) = e^{-q_0 x} g(x).$$

The new equation will be

$$g(x+1) + \frac{1}{e^{q_0}} p(x) g(x) = \frac{x^{-q_1}}{e^{q_0}} \left(1 + \frac{}{x} + \cdots\right).$$

The application of Theorem A then gives

THEOREM 3. *Let p, q, r satisfy hypotheses* (1) *and* (2) *of Theorem* 1, *and in addition assume*

(3) *$q(x)$ is analytic at infinity, with*

$$\lim_{x=\infty} q(x) = q_0, \qquad \lim_{x=\infty} x(q(x) - q_0) = q_1.$$

Then there is a formal solution $\bar{f}(x)$ of (1) *of the form*

$$\bar{f}(x) = e^{-q_0 x} x^{-q_1 - m} \left(f_0 + \frac{f_1}{x} + \cdots\right).$$

Solutions $f_1(x)$ and $f_2(x)$ exist, and have properties similar to those described before.

§ 4.

Case 4. Let $n > 1$. We can now write

$$e^{-\int q(x)dx} = e^{-q^{(1)}(x)} \cdot q_2(x),$$

where

$$q^{(1)}(x) = q_0 \frac{x^{n+1}}{n+1} + \cdots + q_n x,$$

$$q^{(2)}(x) = x^{-q_{n+1}} \left(1 + \frac{q_{n+2}}{x} + \frac{}{x^2} + \cdots\right),$$

the quantities $q_0, q_1, \cdots$ having the significance given in (5). The equation (3) accordingly takes a form to which Theorem A is not applicable. A direct examination must therefore be made in order to determine whether a solution exists.

Consider the associated homogeneous equation

$$g(x+1) + p(x) g(x) = 0.$$

There is a formal solution

$$\bar{g}(x) = x^{mx}(-p_0 e^{-m})^x x^r \left(\bar{g}_0 + \frac{\bar{g}_1}{x} + \cdots\right), \tag{6}$$

*which in general diverges. There are two solutions $g_1(x)$ and $g_2(x)$, analytic in general in the finite plane, and asymptotic to $\bar{g}(x)$, in the right and left half planes, respectively.**

* G. D. Birkhoff, "General Theory of Linear Difference Equations," *Transactions of the American Math. Soc.*, Vol. 12 (1911), pp. 243–284.

Let $g(x)$ represent one (which one to be specified later) of these solutions, and put

$$f(x) = \omega(x)g(x), \tag{7}$$

We then have, upon substituting in (3), the equation

$$\omega(x+1) - \omega(x) = C\frac{e^{-q^{(1)}(x)} \cdot q^{(2)}(x)}{g(x+1)} = G(x). \tag{8}$$

The discussion of this equation falls into various cases.

Case 4a. Suppose $q_0 > 0$. Let $g(x)$ be $g_1(x)$. Then the series

$$\omega(x) = -G(x) - G(x+1) - \cdots \tag{9}$$

is uniformly convergent if the real part of x is positive and sufficiently large; and the series is a solution of (8). The proof in all its details will not be given, but is based upon the following considerations.* If s is sufficiently large, the sth term in (9) can be written, on account of the fact that $g_1(x)$ is asymptotic in the right half plane to $\bar{g}(x)$, ($g(\bar{x})$ having the form given in (6)),

$$\frac{e^{-q^{(1)}(x+s-1)}}{(x+s)^{m(x+s)}(-p_0e^{-m})^{x+s}(x+s)^{r+q_{n+1}}} M(x, s),$$

where $M(x, s)$ is bounded. Now the dominant part of the numerator is, for large s,

$$e^{-\frac{q_0}{n+1}s^{n+1}}$$

and the dominant part of the denominator is $s^{ms+r+q_{n+1}} \cdot e^{-ms}$. The dominant part of the expression is therefore

$$e^{-\frac{q_0}{n+1}s^{n+1}+ms-(ms+r+q_{n+1})\log s}.$$

It is seen that this is the term of a rapidly convergent series, since q_0 is positive. This is true irrespective of the sign of m, for s^{n+1} dominates $s \log s$, for s large, if $n > 0$.

Case 4b. Suppose $q_0 < 0$, and let n be odd. Let $g(x)$ be $g_2(x)$. Then the series

$$\omega(x) = G(x-1) + G(x-2) + \cdots \tag{10}$$

is uniformly convergent for the real part of x negative and sufficiently large; and it is a solution of (8). The proof is effected as before, making use of the fact that $g_2(x)$ is asymptotic to $\bar{g}(x)$ in the left half plane.

Case 4c. Suppose $q_0 < 0$, and let n be even. Neither a series of form (9) nor (10) will converge in this case, with $g(x)$ representing either $g_1(x)$ or $g_2(x)$. This is true because s^{n+1} dominates $s \log s$.

A solution of (8) can be obtained by means of a contour integral due

* Williams, loc. cit., § 3.

to Guichard.* The integral

$$\omega(x) = \int \frac{G(t)dt}{e^{2\pi i(t-x)} - 1}$$

satisfies (8), when the path of integration passes between $x - 1$ and x, with the point x on the right, provided the path extends to infinity in such a way that the integral converges.

In order to determine a choice of the path of integration we shall examine the nature of

$$G(t) = \frac{Ce^{-q^{(1)}(t)} \cdot q^{(2)}(t)}{g(t+1)} = -\frac{Ce^{-q^{(1)}(t)} \cdot q^{(2)}(t)}{p(t)g(t)}.$$

It is obvious that we can neglect $q^{(2)}(t)$ and $p(t)$, if we change the exponent r in the asymptotic form of $\bar{g}(x)$ to $r' = r + m + q_{n+1}$. (This change has the effect of making $q^{(2)}(t)$ and $p(t)$ approach constants as $t = \infty$.)

Let $g(t)$ be $g_1(t)$; then the dominating part in the denominator of $|G(t)|$ is, for t large in the right half plane,

$$|t^{mi}(-p_0e^{-m})^t t^{r'}|.$$

If we put $t = u + iv$, and also $t = \tau e^{i\varphi}$ this becomes

$$e^{(mu+r')\log\tau-(m-\log-p_0)u-m\varphi v} = e^{(\tau m\cos\varphi+r')\log\tau-[(m-\log-p_0)\cos\varphi+m\varphi\sin\varphi]\tau}.$$

Now choose the path of integration so as to make it coincident with the lines

$$\varphi = \pm\frac{\pi}{n+1}$$

at a sufficient distance from $t = 0$. Then $t = \tau e^{i\frac{\pi}{n+1}}$ so that $t^{n+1} = -\tau^{n+1}$. It follows that the dominant part of the real portion of the polynomial $q^{(1)}(t)$ will be, for τ sufficiently great, the positive quantity

$$\frac{|q_0|\tau^{n+1}}{n+1}.$$

Along the contour considered it follows that

$$|G(t)| < M(\tau)e^{-\frac{|q_0|}{n+1}\tau^{n+1}-(\tau m\cos\varphi+r')\log\tau-[(m-\log-p_0)\cos\varphi+m\varphi\sin\varphi]\tau},$$

where $M(\tau)$ is bounded. It is seen that $G(t)$ will approach zero more rapidly than $e^{-\tau}$ as τ approaches infinity, and this irrespective of the value of m, r', and ρ.

A glance will reveal the behavior of the denominator in the Guichard integral on the distant part of the path of integration. Let $x = \xi + i\eta$.

* Williams, loc. cit., § 2.

Then

$$e^{2\pi i(t-x)} - 1 = e^{2\pi i(u-\xi)} \cdot e^{-2\pi(v-\eta)} - 1.$$

It is apparent that along the ray for which $\varphi = \pi/(n+1)$ the quantity just written approaches -1, while along the ray from which $\varphi = -\pi/(n+1)$ it becomes large.

It follows from the considerations given that the integral converges, and will thus furnish a solution of the equation (8).

We shall not inquire into the existence of other solutions or into the behavior of the solution in the infinite part of the plane.

THEOREM 4. *Let p, q, and r satisfy conditions* (1) *and* (2) *of Theorem* 1; *and in addition let $q(x)$ have a pole at infinity. Then it is possible to find a solution of* (8), *such that* (7) *will give a solution of equation* (3), *and therefore of equation* (1).

§ 5.

In the identity (2) we have thus far assumed that $p(x)$ and $q(x)$ were of given form at infinity.

Suppose that $r(x)$ and $p(x)$ are given; then

$$q(x) = \frac{r(x) - p'(x)}{p(x)}.$$

If $p(x)$ is analytic at infinity we may have any of the four cases, depending on the behavior of $r(x)$. If $p(x)$ has a pole at infinity, and $r(x)$ is analytic, or has a pole of lower order than $p(x)$, the equation reduces to Case 2, or possibly Case 1. If $r(x)$ has a pole of the same order as $p(x)$, we have Case 3. If $p(x)$ has a zero at infinity and $r(x)$ does not, the function $q(x)$ will have a pole, and we have Case 4. If $r(x)$ also has a zero at infinity we may have any of the four cases.

Suppose $q(x)$ and $r(x)$ are given. Then

$$p(x) = e^{-\int q(x)dx}\Big[\int e^{\int q(x)dx} r(x)dx + C\Big].$$

Assume that

$$q(x) = x^{n_1}\left(q_0 + \frac{q_1}{x} + \cdots\right),$$

$$r(x) = x^{n_2}\left(r_0 + \frac{r_1}{x} + \cdots\right), \qquad |x| > R.$$

In order that $p(x)$ have the form of a rational function at infinity we must assume that $n_1 < 0$. If $n_1 = -1$, we must in addition assume that q_0 is an integer (positive or negative) such that $q_0 + n_2 \neq -1$. In all cases not excluded equation (1) will come under a form already treated.

BLOOMINGTON, INDIANA,
March, 1921.

THE FOUR COLOR PROBLEM.*

By Philip Franklin.

1. By a map we shall understand a subdivision of an inversion plane or sphere by means of a finite number of circular arcs into a finite number of regions, which completely cover it. There is no loss of generality in this restriction, as a "map" on any surface of genus zero, with a finite number of regions bounded by simple curves, may be deformed into a map of the type just described. A *side* is a line along which two distinct regions touch each other; a *vertex* is a point which belongs to three or more regions. The problem of coloring a map with a given number of colors (denoted in what follows by A, B, C, etc.) is the association of a color with each region in such a way that any two regions with a side in common are given different colors. Two regions with only a vertex (or a finite number of vertices) in common may of course have the same color.

Whether every map can be colored with four colors is an outstanding question, for while no map has ever been exhibited which could not be colored with four colors, no rigorous demonstration of the possibility for the general case has ever been given.† It is known that four colors are necessary to color some maps and that five colors are sufficient to color all maps. If any maps which can not be colored in four colors exist, there must be one such map of a minimum number of regions. We will call such a map an *irreducible* map. It is known that an irreducible map has the following properties:‡

1. Each vertex belongs to three and only three regions.

2. No group of less than five regions forms a multiply-connected portion of the map. (Consequently there are no two-, three- or four-sided regions and no multiply-connected regions.)

3. No group of five regions forms a multiply-connected portion of the map unless the group consists of the five regions surrounding a pentagon.

4. No edge is surrounded by four pentagons.

5. No region is completely surrounded by pentagons.

* Presented to the National Academy of Sciences, November 17, 1920.

† A history of the question, with a bibliography, is given in the thesis of Alfred Errera "Du Coloriage des Cartes, etc.," Brussels, 1921.

‡ For an account of these reductions of the problem, which are due to A. B. Kempe and G. D. Birkhoff, see a paper by the latter "The Reducibility of Maps," American Journal of Mathematics, Vol. XXXV (1913), p. 116.

6. No even-sided region is completely surrounded by hexagons. Each of these statements amounts to saying that a certain configuration is not possible in an irreducible map. Such a configuration will be called a *reducible* configuration.

In this paper we shall derive a few additional reducible configurations, by means of which it can be shown that the number of regions in an irreducible map is greater than 25. An example is also given of a map of 42 regions which, although colorable in four colors, satisfies all the conditions which have been derived for an irreducible map. It may be taken as showing the extreme lack of generality of the results thus far obtained for this problem.

2. It is interesting to find out some of the properties of a map not containing any region of less than five sides, as this is a property of an irreducible map. Since our map is drawn on a sphere, the Euler formula (applied to a manifold of genus zero) gives:

$$a_0 - a_1 + a_2 = 2, \tag{1}$$

where a_0, a_1, a_2 are the number of vertices, sides and regions respectively. Also since only three regions touch any one vertex, we have:

$$2a_1 = 3a_0 = \sum_5 \nu A_\nu, \tag{2}$$

where A_ν means the number of regions of ν sides in the map; the last two expressions are each equal to the first since they represent twice the number of sides in the map, counted first with reference to vertices, then with reference to regions. From (1) and (2) we obtain:

$$a_1 = 3(a_2 - 2), \qquad a_0 = 2(a_2 - 2). \tag{3}$$

From (2), (3) and the fact that $a_2 = \sum_5 A_\nu$, we see that

$$6(\sum_5 A_\nu - 2) = \sum_5 \nu A_\nu. \tag{4}$$

This may be written:

$$A_5 = 12 + \sum_7 (\nu - 6)A_\nu \tag{5}$$

and since the second term on the right is positive, A_5 must be at least 12, and we have the well-known theorem:

*Every map containing no triangles or quadrilaterals and having three regions abutting on each vertex contains at least twelve pentagons.**

* Cf. Kempe, A. B., "The Geographical Problem of the Four Colors," AMERICAN JOURNAL OF MATHEMATICS, Vol. II (1879), p. 198.

We shall also prove that such a map must contain either:

A pentagon adjacent to two other pentagons,
A pentagon adjacent to a pentagon and to a hexagon, or
*A pentagon adjacent to two hexagons.**

For, consider a map with none of these combinations of regions and let us count the number of vertices in the map which belong to a hexagon or a pentagon. We find that the number of vertices contributed by hexagons nowhere in contact with pentagons will be greater than twice the number of such hexagons since each hexagon has six vertices and no vertex belongs to more than three hexagons; pentagons isolated from hexagons or other pentagons will give five vertices each; two pentagons adjacent to each other but to no other pentagons or hexagons give eight vertices together, and hence average four each; while a pentagon adjacent to a hexagon gives over four vertices, since of its five vertices we need only deduct two thirds to account for the two where the hexagon joins it. Thus if none of the three conditions enumerated above existed, the number of vertices would be at least $4A_5 + 2A_6$. That is we would have to have:

$$a_0 \geq 4A_5 + 2A_6. \tag{6}$$

But from (5) and the obvious inequality:

$$0 \geq \sum_7 (7 - \nu)A_\nu \tag{7}$$

there results:

$$\sum_7 A_\nu + 12 \leq A_5 \tag{8}$$

or

$$\sum_5 A_\nu + 12 \leq 2A_5 + A_6 \tag{9}$$

and since (from (3)):

$$\sum_5 A_\nu = a_2 = a_0/2 + 2, \tag{10}$$

$$a_0/2 + 14 \leq 2A_5 + A_6, \tag{11}$$

$$a_0 + 28 \leq 4A_5 + 2A_6, \tag{12}$$

which contradicts (6) and thus proves the theorem.

The above theorem is not restricted to irreducible maps, but it follows from it that if the configurations there shown to be present were reducible there could not be any irreducible maps and the four-color problem would be solved. While it does not appear to be possible to prove this, there are a number of more complicated configurations which are reducible.

* That every such map contains either two adjacent pentagons or a pentagon adjacent to a hexagon was proved by Wernicke, P., "Über den Kartographischen Vierfarbensatz, Mathematische Annalen," Vol. 58 (1904), p. 419.

3. To obtain these configurations, and prove their reducibility, we shall need the notion of *chains*, originated by Kempe,* and employed by Birkhoff.* If a map is colored, or partially colored, a group of regions colored in two colors (say A and B), forming a connected region, and such that each region adjacent to a region of the group is either colored in one of the remaining two colors (C or D) or not yet colored, is said to form a *chain* (an AB chain). Evidently we may obtain a second coloration or partial coloration of the map by interchanging the two colors on a single chain, and unless the map contains only one chain in this pair of colors, the new coloration will differ from the old by more than a mere permutation of the colors of the whole map.

Furthermore since two chains with no color in common, as an AB chain and a CD chain, can not "cross" each other, if we have a closed circuit consisting of an AB chain, or an AB chain and an uncolored region in the case of a partially colored map, it follows that the C and D regions on one side of the closed circuit can not belong to the same CD chain as those on the opposite side of the circuit. Thus in Fig. 1, if 1 is an uncolored region,

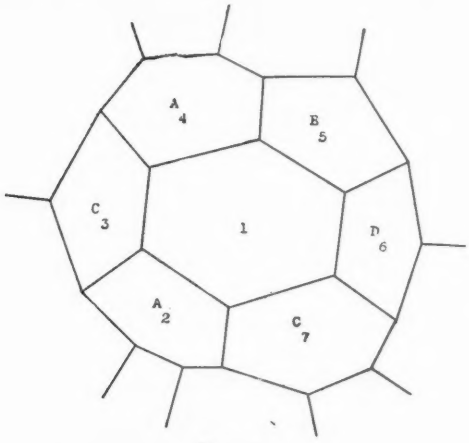

FIG. 1.

and 2 and 4 are joined by an AB chain, 3 must belong to a CD chain distinct from the one containing 6 and 7. Consequently we may interchange the colors in the CD chain containing 3 without affecting 6 and 7. Since, in most of the applications of this process, we shall only be concerned with the arrangement of the colors about the uncolored region, and the rest of them are unchanged by this operation, we shall briefly refer to this operation as "changing 3 to a D." The value of these operations will be seen in the proofs which follow.

* L. c.

4. We shall now prove that

A side of a hexagon surrounded by this hexagon and three pentagons is a reducible configuration. For, if it were present in an irreducible map, and we erased the dotted lines as in Fig. 2, we would obtain a new map which

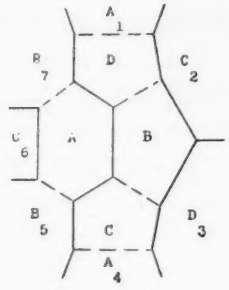

FIG. 2 *a*.

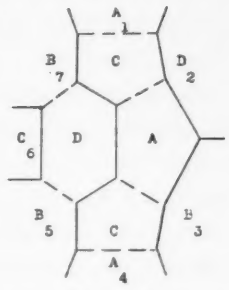

FIG. 2 *b*.

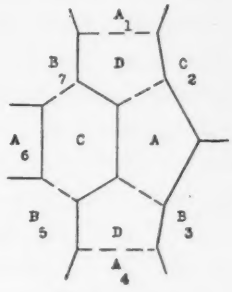

FIG. 2 *c*.

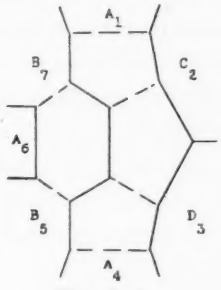

FIG. 2 *d*.

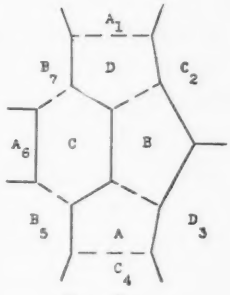

FIG. 2 *e*.

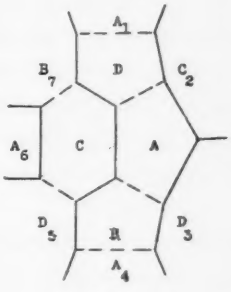

FIG. 2 *f*.

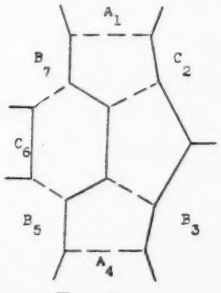

FIG. 2 *g*.

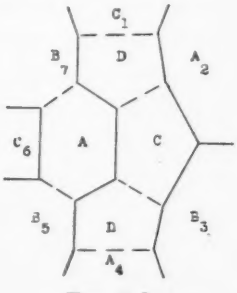

FIG. 2 *h*.

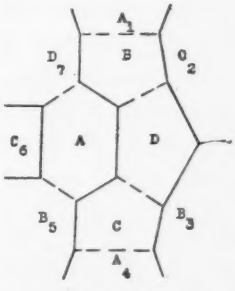

FIG. 2 *i*.

would contain fewer regions than an irreducible map and hence be colorable. From the way we selected the lines which were erased, regions 1 and 4 would have the same color (say *A*) while regions 5 and 7 would have a different common color (say *B*). Of the five essentially distinct colorations, the three cases shown in 2 *a*, 2 *b* and 2 *c* permit of immediate coloration, as indicated. In the case shown in 2 *d*, if 5 is joined to 7 by a *BD* chain, 6 may be changed to a *C*, reducing the problem to case 2 *a*, while if 5 is

joined to 3 by a *BD* chain, 4 may be changed to a *C*, and the map colored as shown in 2 *e*. If neither of these chains exist, 5 may be changed to a *D*, and the map colored as shown in 2 *f*. Finally in the case given in 2 *g*, either a *BD* chain joins 7 with 5, and we reduce to 2 *c* by changing 6 to *A*; or a *BD* chain joins 7 with 3, and we color as in 2 *h* after interchanging *A* and *C* in the *AC* chain including 1 and 2; or 7 may be changed to a *D* and we color as in 2 *i*.

If 5 and 7 had a side in common in our original map, we could not erase the dotted lines, and still leave a map; but in this case we would have three regions forming a multiply-connected piece. If 1 and 4 had a side in common, we would have a group of five regions forming a multiply-connected region of the map, and not all adjacent to the same pentagon. Hence both these cases are excluded by the properties of irreducible maps given in the first section.

If a pentagon is in contact with three pentagons, a hexagon, and a fifth region of any number of sides, either the hexagon is adjacent to this fifth region, in which case the three adjacent pentagons, with the initial pentagon, completely surround an edge, or the hexagon is adjacent to two pentagons, and we have a side of the hexagon completely surrounded by this hexagon and three pentagons. In either case, it is reducible and we have the result:

A pentagon in contact with three pentagons and a hexagon is a reducible configuration.

Also if a pentagon is in contact with two pentagons and three hexagons, if the two pentagons are not adjacent, they are separated by a hexagon, which with the three pentagons forms a reducible configuration. If the two pentagons are adjacent, we proceed as follows: We erase the boundaries which are dotted (Fig. 3 *a*) and color the resulting map. If all the regions

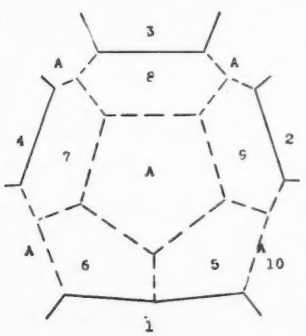

FIG. 3 *a*.

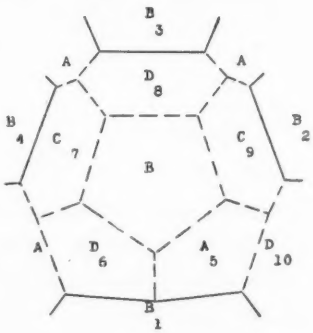

FIG. 3 *b*.

1, 2, 3, 4 are not colored in one cc or, there are two of them, say 1 and 2,

which are separated by a single region and in different colors, say B and C respectively. We then color 5 with C, and 6, 7, 8, 9 in turn, which is possible since taking them in this order we shall never come to a region surrounded by more than three different colors. On the other hand, if all the regions 1, 2, 3, 4 are colored in the same color, say B, either there is no BC chain joining all these regions, in which case we can change some of these regions to C and reduce our problem to the case just discussed, or the AD chain containing the region 10 is separated from the other regions marked A and can be changed to a D. The map is then colored as shown in Fig. 3 *b*. This proves the theorem:

A pentagon surrounded by two pentagons and three hexagons is a reducible configuration.

In this proof we have omitted any reference to the case where the dotted lines can not be erased without giving rise to a region which meets itself along one edge. We shall also do this in future cases where, as in this case, it may be excluded by the considerations used for this purpose in the proof of our first theorem.

By a method quite similar to the above, we could easily show that any odd-sided region, completely surrounded by one or more pairs of pentagons, the two of each pair being adjacent, and a number (necessarily odd) of hexagons, is a reducible configuration.

To lead up to a slightly more general theorem, we repeat Birkhoff's proof of the reducibility of an even-sided region surrounded by hexagons, for definiteness stating the proof for a hexagon so surrounded. We erase the dotted lines of Fig. 4 *a* and obtain the coloration shown. If all the

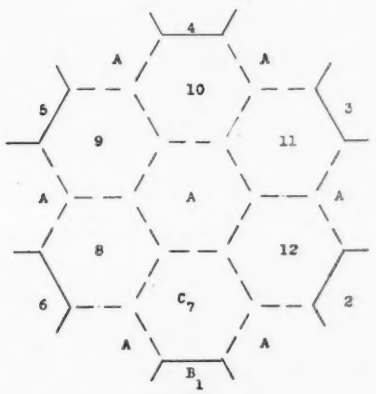

FIG. 4 *a*.

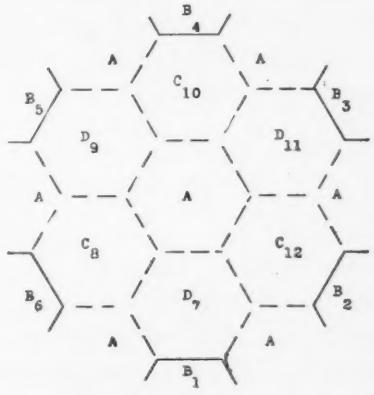

FIG. 4 *b*.

regions 1, 2, 3, 4, 5, 6 are not colored in one color, there are two of them, say 1 and 2, which are separated by a single region and in different colors,

say B and C. We then color 7 with C, and color 8, 9, 10, 11, 12 in turn, which is possible since we shall find each adjacent to regions of three different colors at most, and thus have a fourth with which to color each. If 1, 2, 3, 4, 5, 6 are all in the same color, our map is colored as in Fig. 4 *b*.

Our generalization is to the case where two adjacent hexagons are replaced by pentagons, and the above method is directly applicable, provided we imagine one of the regions marked A as shrunk to a point. This shows that:

An even-sided region completely surrounded by hexagons and pairs of pentagons, the two of each pair being adjacent, is a reducible configuration.

If a hexagon is surrounded by two hexagons and four pentagons, either the pentagons are grouped so as to come under the theorem just proved, or one of the edges of the hexagon is in contact with three pentagons which we proved above was a reducible configuration. Thus:

A hexagon surrounded by four pentagons and two hexagons is a reducible configuration.

A region of an even number of sides $(2n)$ *surrounded by* $2n - 2$ *pentagons and two other regions, which are adjacent, is reducible.*

To fix the ideas, we state the proof for an octagon. We erase the dotted lines (Fig. 5) and color the resulting map. If 4 is a C, we color 15, 14, 13,

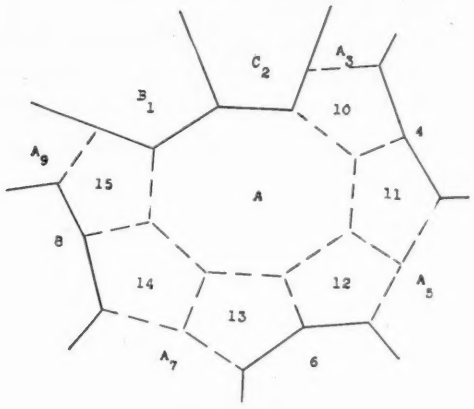

FIG. 5.

12, 11, 10 in turn which will be possible since each will only be adjacent to regions in at most three different colors when we come to it. If 4 is a B or D, we color 10 D or B respectively, and color 11 C. If 6 is a C, we color 15, 14, 13, 12 in turn, as before; while if it is B or D we color 12 D or B respectively and 13 C. We then color 14 and 15 D, C; B, D; or B, C; according as 8 is B, C or D.

A region of an odd number of sides $(2n - 1)$ *surrounded by* $2n - 2$ *pentagons and one other region is reducible.*

We give the proof for a heptagon. After erasing the dotted lines (Fig. 6) we color the new map, as indicated. Reasoning exactly as for the

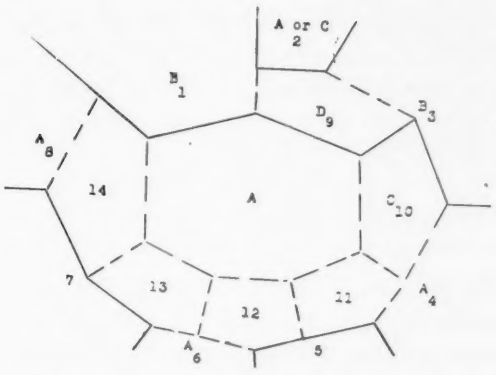

FIG. 6.

preceding theorem, we show that if 5 is a C the map is colorable, while if it is not C we color 11 and 12, giving 12 the color C. Then we color 13 and 14 D, C; B, D; or B, C; according as 7 is B, C or D.

It follows from our last two theorems that:

An n-gon in contact with $n - 1$ pentagons is reducible.

5. We will now deduce certain inequalities which must be satisfied by the numbers A_ν if the map is an irreducible one. We have shown that such a map must not contain:

(a) An n-gon in contact with $n - 1$ pentagons,

(b) A pentagon in contact with three pentagons and one hexagon,

(c) A pentagon in contact with two pentagons and three hexagons,

(d) A hexagon in contact with four pentagons and two hexagons,

in addition to the six configurations given in the first section, and have also shown that the equation (see (5)):

$$A_5 = 12 + \sum_7 (\nu - 6)A_\nu \tag{13}$$

applies to such a map.

From (a) we know that every region of our map is in contact with at least *two* regions of more than five sides. Hence the number of sides (a side being counted with each of the two regions it separates) of regions with more than five sides must be at least equal to twice the total number of regions. That is:

$$\sum_6 \nu A_\nu \geq 2 \sum_5 A_\nu. \tag{14}$$

To obtain a second inequality from the remaining conditions, we write:

A_5^0 = the number of pentagons in contact with no region of more than six sides,
A_5^1 = the number of pentagons in contact with only one such region,
A_5^2 = the number of pentagons in contact with two or more such regions,
A_6^0 = the number of hexagons in contact with no such regions, and
A_6^1 = the number of hexagons in contact with at least one such region.

From (*a*) it follows that each region A_5^2, A_6^1 (we thus abbreviate regions of the type counted in A_5^2, A_6^1) as well as those of more than six sides is in contact with at least *two* regions of more than five sides. Also from (*a*), (*b*) and (*c*) it follows that each pentagon A_5^0 is in contact with at least *four* hexagons, and from (*a*) and (*b*) that each pentagon A_5^1 is in contact with at least two hexagons in addition to the one region of more than six sides; and therefore to at least *three* regions of more than five sides. Finally from (*a*) and (*d*) we see that each hexagon A_6^0 is in contact with at least *three* other hexagons. Thus we have:

$$\sum_6 \nu A_\nu \geq 4A_5^0 + 3A_5^1 + 2A_5^2 + 3A_6^0 + 2A_6^1 + 2\sum_7 A_\nu. \tag{15}$$

But from the definitions of the regions A_5^1, etc.:

$$\sum_7 \nu A_\nu \geq A_5^1 + 2A_5^2 + A_6^1. \tag{16}$$

If we add corresponding members of (15) and (16), recollecting that $A_5 = A_5^0 + A_5^1 + A_5^2$ and $A_6 = A_6^0 + A_6^1$, we obtain the result:

$$\sum_6 \nu A_\nu + \sum_7 \nu A_\nu \geq 4A_5 + 3A_6 + 2\sum_7 A_\nu, \tag{17}$$

which may be written:

$$2\sum_7 \nu A_\nu \geq 4A_5 - 3A_6 + 2\sum_7 A_\nu. \tag{18}$$

For a map which contains no regions of more than seven sides, we may obtain a somewhat stronger inequality, by using:

$$4A_6 + 5A_7 \geq 4A_5^0 + 3A_5^1 + 2A_5^2, \tag{19}$$
$$5A_7 \geq A_5^1 + 2A_5^2, \tag{20}$$

which are analogous to (15) and (16); except that by considering only sides of pentagons in contact with regions of more than five sides we are enabled to use (*a*) in deriving the left members. They give:

$$4A_6 + 10A_7 \geq 4A_5, \tag{21}$$

which is only applicable to maps composed entirely of pentagons, hexagons, and heptagons.

By using (13) we may reduce (14) and (18) to the respective forms:

$$\sum_{6} (10 - \nu)A_\nu \geq 24, \tag{22}$$

$$2\sum_{7} (11 - \nu)A_\nu \geq 48 - 3A_6. \tag{23}$$

These two inequalities show that the map we are considering must have at least 25 regions, and if it have only 25, they must be 17 pentagons, 3 hexagons, and 5 heptagons. For if the map contained two hexagons (22) would give:

$$\sum_{7} (10 - \nu)A_\nu \geq 24 - 4A_6 = 16, \tag{24}$$

which requires at least six regions of more than six sides, and hence by (13) at least 18 pentagons. This would make 26 regions. If the map contained less than two hexagons, the same equation would show that there were more than 27 regions in the map, by a similar argument. Also our map can not have less than five regions of more than six sides, since if the map contained four heptagons, (23) would give:

$$3A_6 > 16, \tag{25}$$

and the six hexagons required by (25) together with the 16 pentagons required by (13) would make 26 regions. Furthermore each heptagon less than four will increase the right member of (25) by eight, and hence require at least two additional hexagons in place of the heptagon and pentagon removed. Replacing any of the heptagons by octagons or regions of more than eight sides will strengthen our inequalities, as well as necessitating more pentagons to satisfy (13).

But the map of 17 pentagons, three hexagons, and five heptagons is not irreducible, since it does not satisfy (21). Consequently every irreducible map must contain more than 25 regions and this gives the theorem:

Every map containing 25 *or fewer regions can be colored in four colors.*

6. The question naturally arises whether 25 is the greatest number for which we can prove such a theorem as the above on the basis of the reductions already described. While an exact answer to this question is lacking, it is evident that the smallest number of regions in a map not containing any of these known reducible configurations is not considerably above 25, as we can construct a map with a small number of regions not containing any of them. Thus in Fig. 7 we exhibit a map of 42 regions which satisfies all the properties of irreducible maps given by previous writers as well as those derived in this paper. The map may be formed by constructing a hexagon on each of the 30 edges of a regular dodecahedron, in such a way

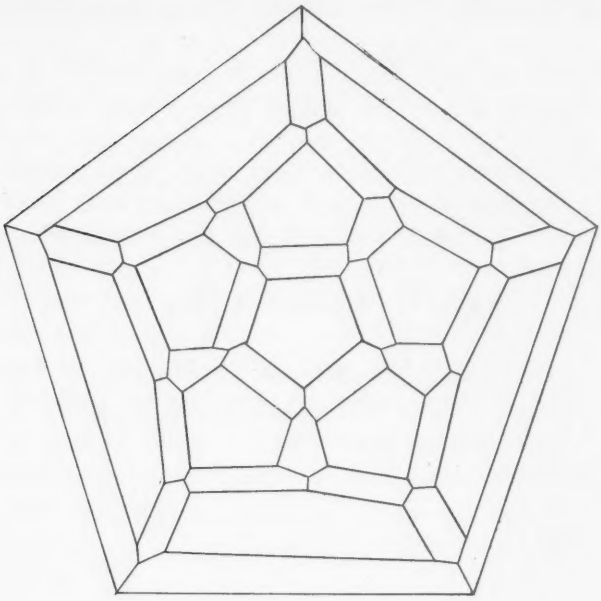

FIG. 7.

as to leave twelve pentagonal faces,* and may be colored by first coloring the pentagons as they would be colored for the dodecahedron.

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* This process was suggested to the writer in another connection by Prof. J. W. Alexander.